Edge-Erdős-Pósa through packing and contraction

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Université Libre de Bruxelles

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The classic EP-theorem

Theorem (Erdős and Pósa, 1965)

There is a function $f(k) = O(k \log(k))$ s.t. for every graph G and $k \in \mathbb{N}$

1. G contains k vertex-disjoint cycles, or
2. there exists $X \subseteq V(G)$, $|X| \leq f(k)$ s.t. $G - X$ has no cycles.

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Vertex-Erdős-Pósa property

A class of graphs $\mathcal{H}$ has the *vertex-EP property* with *bounding function* $f : \mathbb{N} \to \mathbb{R}$ if for every graph G and every $k \in \mathbb{N}$,

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Analogue for the edge-EP property?

No (clear) analogue for edge-EP property

Theorem, Bruhn, Heinlein and Joos (2018+)

There exist planar graphs J such that $\mathcal{M}(J)$ does not have the edge-EP property, e.g. if J is:

- a ladder of length at least 71, or
- a binary tree of height at least 37.

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$\mathcal{M}(J)$ does have the edge-EP property if $J = \ldots$

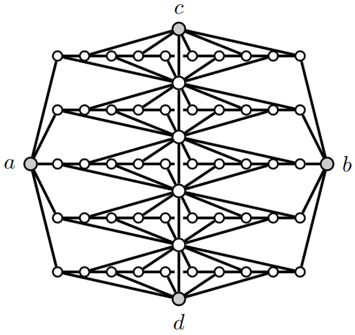
- C_3 (classic EP-theorem, 1965)
- C_l, for any $l \geq 3$ ('long cycles', Bruhn, Heinlein and Joos, 2019)
- θ_l, for any $l \geq 1$ (Chatzidimitriou, Raymond, Sau, Thilikos, 2018)
- K_4 (Bruhn and Heinlein, 2019+)
- ?

Conjectures for edge-EP

Conjecture (Bruhn, Heinlein and Joos 2019+)

$\mathcal{M}(J)$ does not have the edge-EP property if J is a planar graph with sufficiently large treewidth.

Conjectures for edge-EP



Conjecture (Bruhn, Heinlein and Joos 2019+): if J is a planar graph such that some sufficiently large ‘condensed wall’ contains a J-expansion, then $\mathcal{M}(J)$ has the edge-EP property.

Long cycles

Theorem (Bruhn and Heinlein, 2019)

For every fixed integer $l \geq 3$ and every $k \in \mathbb{N}$, every graph G contains

- k edge-disjoint cycles of length at least l ('long cycles') or
- an edge set of size $O(k^2 \cdot \log k + lk)$ such that $G - X$ has no long cycles.

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The same, but instead with $O(lk \cdot \log(lk))$.

Compare with:

Theorem (Moesset, Noever, Skorić and Weissenberger, 2016+)

The vertex-EP property for long cycles holds with (optimal) bounding function $O(k \cdot \log k + lk)$.

New proof of the classic Erdős-Pósa theorem

$$g(k) := 8\log_2(k+1) + 2$$

Classic edge-EP

For every graph G and every $k \in \mathbb{N}$

1. G contains k edge-disjoint cycles, or
2. there exists a edge set X of size at most $k \cdot g(k)$ s.t. $G - X$ has no cycles.

Remark: the proof for vertex-EP is the same

Plan of the proof

$$g(k) := 8\log_2(k+1) + 2$$

1. By induction on $|E(G)|$, may assume that
   - girth is larger than $g(k)$, and
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2. $\Rightarrow$ G has a minor G' with minimum degree $\geq 3k$.
3. $\Rightarrow$ G' contains k vertex-disjoint cycles.
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Proof that we may assume girth $> g(k)$

Suppose that G has girth $\leq g(k)$.

Let C be a shortest cycle and apply induction to $G^* = G - E(C)$.

1. Either G^* has $k-1$ edge-disjoint cycles, or
2. $G^* - X^*$ has no cycles, for some edge set X^* of size at most $(k-1) \cdot g(k-1)$.

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Suppose that G has girth $\leq g(k)$.

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1. If G^* has $k-1$ edge-disjoint cycles, then G has k edge-disjoint cycles.

:-)

Proof that we may assume girth $> g(k)$

Suppose that G has girth $\leq g(k)$.

Let C be a shortest cycle and apply induction to $G^* = G - E(C)$.

1. If G^* has $k-1$ edge-disjoint cycles, then G has k edge-disjoint cycles.
2. If $G^* - X^*$ has no cycles, for some edge set X^* of size at most $(k-1) \cdot g(k-1)$, then set $X := X^* \cup E(C)$.
   Then $G - X$ has no cycles and

$$\begin{aligned} |X| &= |X^*| + |E(C)| \\ &\leq (k-1) \cdot g(k-1) + g(k) \\ &\leq k \cdot g(k). \end{aligned}$$

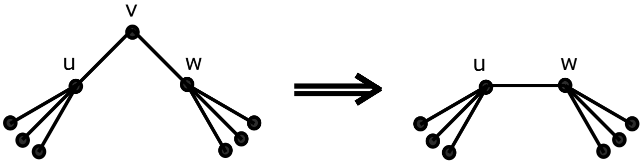
:-)

Proof that we may assume minimum degree ≥ 3

Vertices of degree 0 or 1 are not visited by any cycle, so may assume minimum degree ≥ 2.

Proof that we may assume minimum degree ≥ 3

- If $\deg(v) = 2$ for some vertex v, then consider a neighbour u of v. Contract edge uv. Apply induction to the resulting graph G^*.



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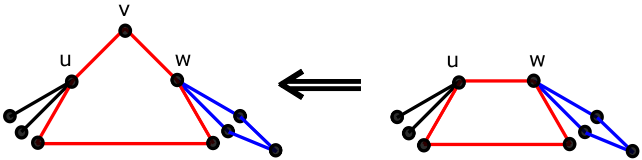

G has k edge-disjoint cycles $\Leftarrow$ G^* has k edge-disjoint cycles.

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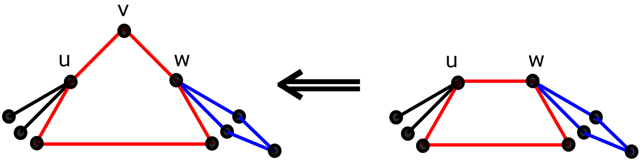

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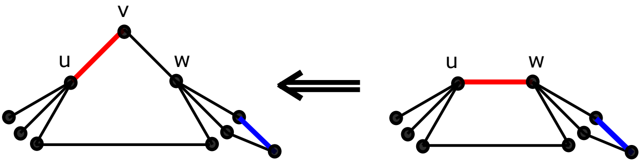

$G - X$ has no cycles $\Leftarrow$ $G^* - X^*$ has no cycles.

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   - girth is larger than $g(k)$, and
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2. $\Rightarrow$ *G has a minor G' with minimum degree* $\geq 3 \cdot 2^{g(k)/8} \geq 3k$.
3. $\Rightarrow$ G' contains k vertex-disjoint cycles.
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Find minor with minimum degree $\geq 3 \cdot 2^{g(k)/8} \geq 3k$

Recall: girth $\geq g(k)$ and minimum degree ≥ 3.

Could directly apply

Lemma, Kühn and Osthus (2003)

Let $l \geq 1$ and $g \geq 3$ be integers. Then every graph of girth at least $8l + 3$ and minimum degree r contains a minor of minimum degree at least $r(r-1)^l$.

but let's prove it.

Find minor with minimum degree $\geq 3 \cdot 2^{g(k)/8} \geq 3k$

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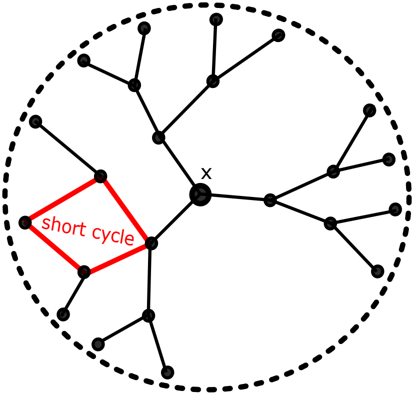
Consider a maximal packing of G with balls of radius $r := g(k)/8$.

- No ball contains a cycle.



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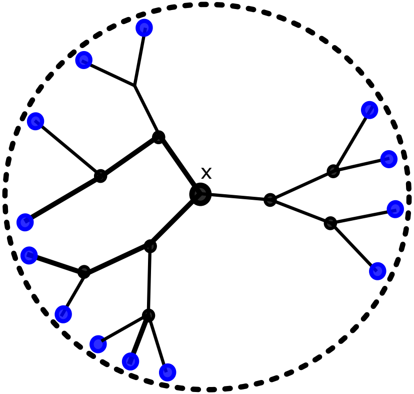

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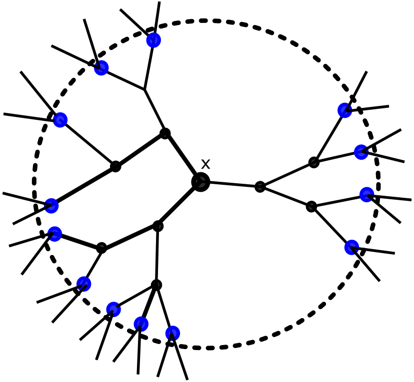

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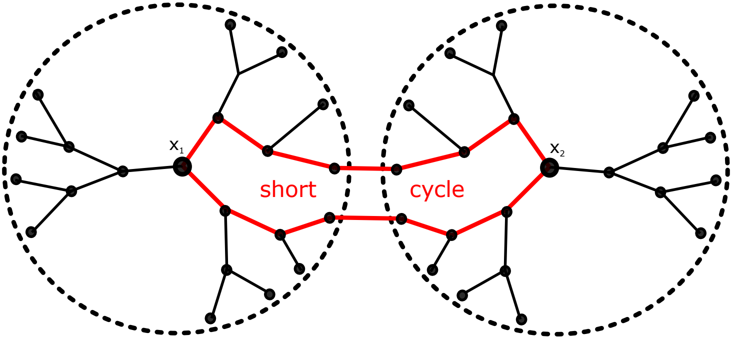

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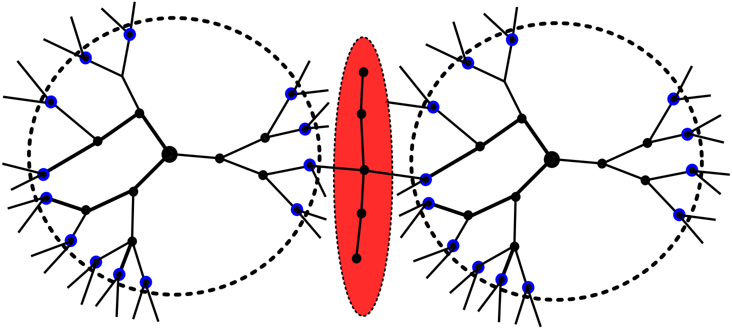
- Every leaf of a ball has neighbours in some other ball
- Each ball induces a tree with at least $3 \cdot 2^{r-1}$ leaves.
- Every two balls are joined by at most one edge.
- Contract each ball $\rightarrow$ minor G' with minimum degree at least $3 \cdot 2^r$.

If the balls do not cover $V(G)$



Neighbours of leaves are not necessarily in another ball.

Find minor with minimum degree $\geq 3 \cdot 2^{g(k)/8} \geq 3k$

Instead:
Let X be a maximal set of vertices that are pairwise at distance at least $g(k)/4$.

- Consider balls $(B(x))_{x \in X}$ of radius $r := g(k)/8$.
- Add each vertex at distance $r+1$ from X to one of the balls it is adjacent to.
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- etc. until every vertex is covered.

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Yields a partition of $V(G)$ with approximate 'balls' $(H(x))_{x \in X}$ such that each $v \in H(x)$ is at distance at most $2r$ from x.

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Plan of the proof

$$g(k) := 8\log_2(k+1) + 2$$

1. By induction on $|E(G)|$, may assume that
   - girth is larger than $g(k)$, and
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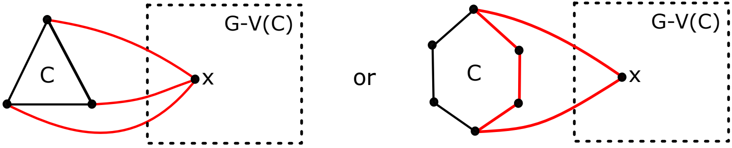

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- Induction: $G - V(C)$ has $k-1$ vertex-disjoint cycles $\{C_1, \ldots, C_{k-1}\}$.
- So G has k vertex-disjoint cycles $\{C, C_1, \ldots, C_{k-1}\}$.

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:-)

Plan of the proof; how to adapt to long cycles?

$$g(k) := C \cdot \log_2(k+1)$$

- By induction on $|E(G)|$, may assume that
  - each cycle has length larger than $g(k)$, and
  - minimum degree is at least 3.
- Partition $V(G)$ into approximate balls of radius $r := g(k)/8$.
- Each ball induces a tree with $2^{\Omega(r)}$ leaves. (otherwise short cycle)
- Every two balls are joined by at most one edge. (otherwise short cycle)
- Contract each ball $\rightarrow$ minor G' with minimum degree $\geq 2^{\Omega r} = \Omega(k)$.
- $\Rightarrow G'$ contains k vertex-disjoint cycles.
- $\Rightarrow G$ contains k edge-disjoint cycles.

Plan of the proof; how to adapt to long cycles

$$g(k) := C \cdot l \cdot \log_2((k+1)(l+1))$$

- By induction on $|E(G)|$, may assume that
  - each long cycle has length larger than $g(k)$, and
  - 'locally', each block is incident to at least three other blocks.
- Partition $V(G)$ into approximate balls of radius $r := g(k)/8$.
- Each ball induces a tree-like graph with $2^{\Omega(r)}$ leaf-blocks. (otherwise short long cycle)
- Edges between two distinct balls are locally concentrated (essentially only between two leaf-blocks). (otherwise short long cycle)
- Contract each ball → minor G' with minimum degree $\geq 2^{\Omega r} = \Omega(k)$.
- $\Rightarrow G'$ contains k vertex-disjoint long cycles.
- $\Rightarrow G$ contains k edge-disjoint long cycles.

More general scheme?

Suppose we want to prove the edge-EP property for a graph class $\mathcal{H}$, with some bounding function $k \cdot g(k) = O(k \log k)$.

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  - each copy of $H \in \mathcal{H}$ in G has more than $g(k)$ edges.
- Partition $V(G)$ into approximate balls of radius $O(g(k))$.
- Show that each ball exhibits some form of exponential growth. Ie, has $2^{\Omega(g(k))}$ vertices that have a neighbour in some other ball.

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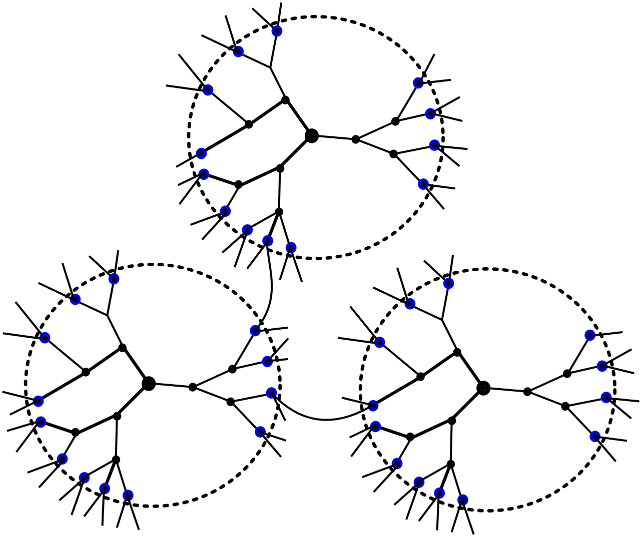
- Characterize the planar graphs J for which $\mathcal{M}(J)$ has the edge-EP property.

Thank you for your attention!



Sketch of the proof

- By induction on $|V(G)|$, may assume that girth$\geq g(k)$.
- Cover $V(G)$ with vertex-disjoint balls of radius $r := g(k)/8$.
- derive that G has minimum degree 3
- show that each ball has $e^{\Omega(r)}$ leaves.
- every two balls are joined by at most one edge (otherwise short cycle)
- contract each ball $\to$ minor G' with minimum degree $e^{\Omega(r)} \geq 3k$.
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