The list-packing number of graphs

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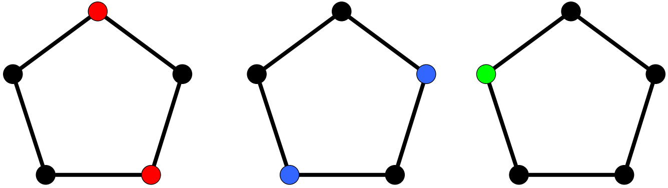
Joint work with Stijn Cambie, Ewan Davies and Ross Kang

TU Delft, 1 april 2022

Independent set: subset of vertices without any edge between them.



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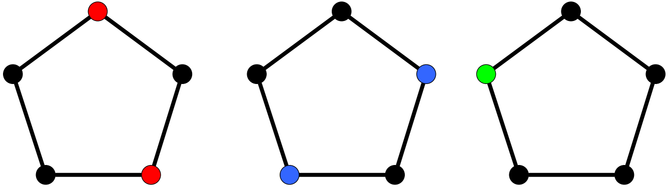
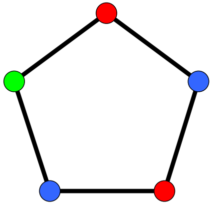


Proper colouring: partition into independent sets



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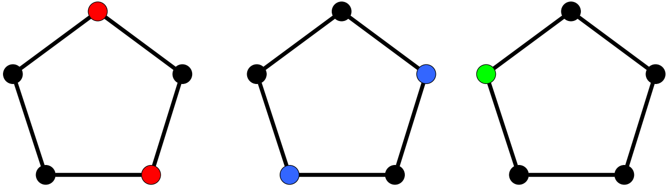

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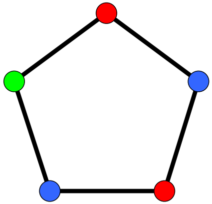
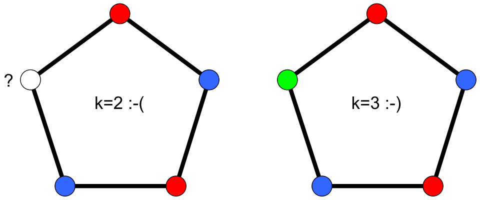


Equivalent: adjacent vertices must have distinct colours.

A (proper) *k-colouring* is a partition of $V(G)$ into $\leq k$ independent sets



Chromatic number $\chi(G)$: smallest integer k such that G admits a k-colouring.

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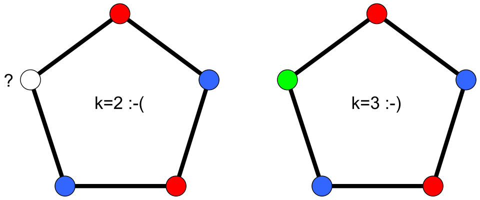


Chromatic number $\chi(G)$: smallest integer k such that G admits a k-colouring.

Example

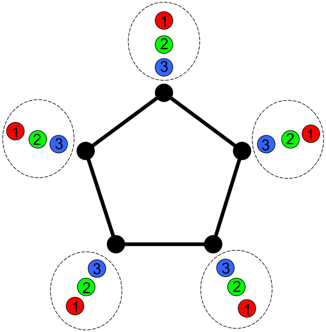
If G is a cycle of length n, then $\chi(G) = 2$ if n even, $\chi(G) = 3$ if n odd.

Thank you for your attention!

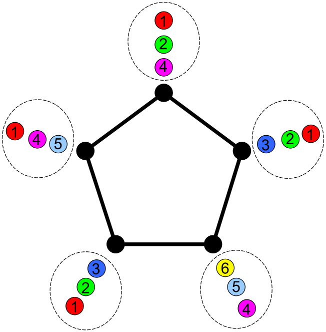
1 april!

In usual graph colouring: each vertex chooses one colour from a common set $[k] := \{1, 2, \ldots, k\}$.

In *list-colouring*, each vertex v has a private list of colours $L(v) \subseteq \mathbb{N}$ to choose one colour from.



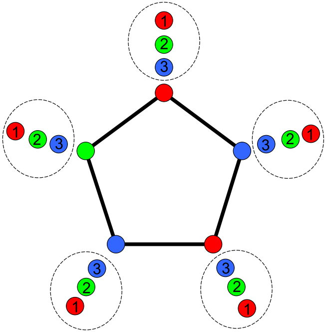
same lists



distinct lists possible

In usual graph colouring: each vertex chooses one colour from a common set $[k] := \{1, 2, \ldots, k\}$.

In *list-colouring*, each vertex v chooses one colour from a private list $L(v) \subseteq \mathbb{N}$ of size k.



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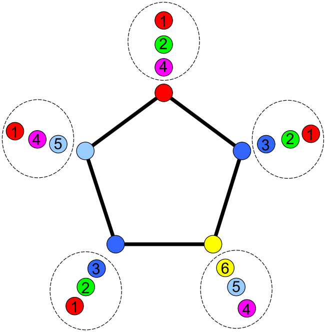

distinct lists possible

A *k-list-assignment* is a mapping $L : V(G) \to \mathbb{N}$ such that $|L(v)| = k$ for each vertex v.

An L-colouring is a proper colouring such that each vertex v receives a colour from its list $L(v)$.

The *list-chromatic number* $\chi_\ell(G)$ is the smallest integer k such that *every* k-list-assignment L admits an L-colouring.

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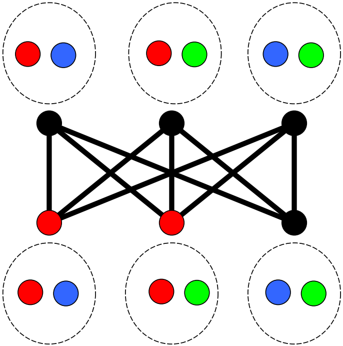
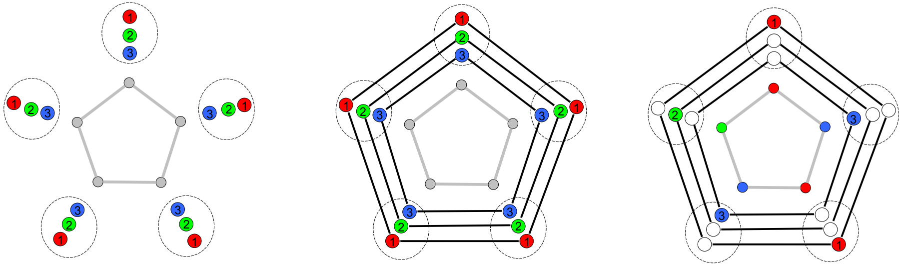


A 2-list-assignment of a bipartite graph that does not admit a colouring. So $\chi_\ell(G) > 2$.

List-colouring rephrased as an independent set problem

An L-colouring of G is equivalent to an independent set of size $|V(G)|$ in the *cover graph* $\mathcal{B}_L(G)$:

Blow-up each vertex v to a clique on $L(v)$. Then add an edge between (v, c_1) and (u, c_2) in $\mathcal{B}_L(G)$ iff $uv \in E(G)$ and the colours c_1, c_2 are equal.



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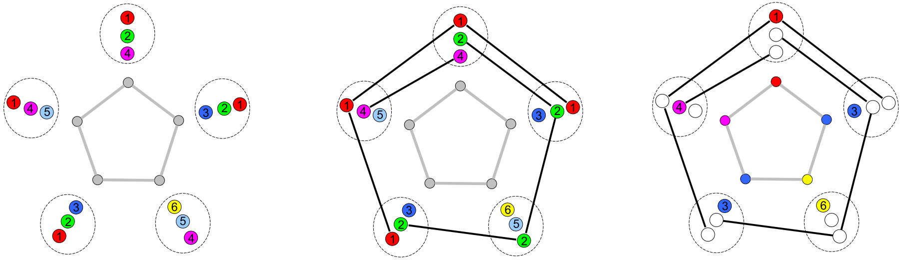


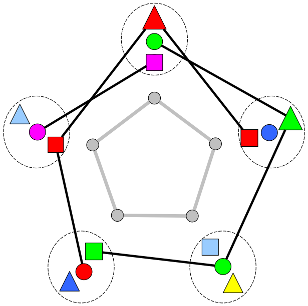
G with lists L $\qquad$ $\mathcal{B}_L(G)$ $\qquad$ indep. set in $\mathcal{B}_L(G)$

Next step: Instead of one L-colouring, we wish to find many L-colourings in parallel.

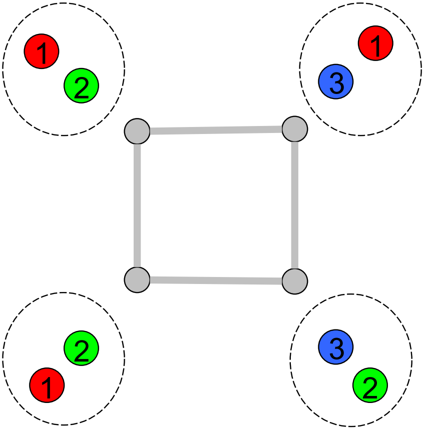
Definition

Given a k-list-assignment L, an *L-packing* is a collection of k disjoint L-colourings.

Equivalently: a *partition* of $\mathcal{B}_L(G)$ into k independent sets of size $|V(G)|$.



Example: a 2-list assignment of C_4 that does not admit a packing.



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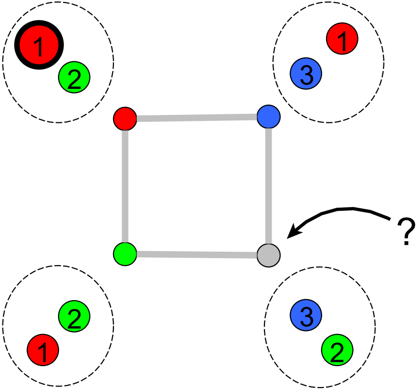

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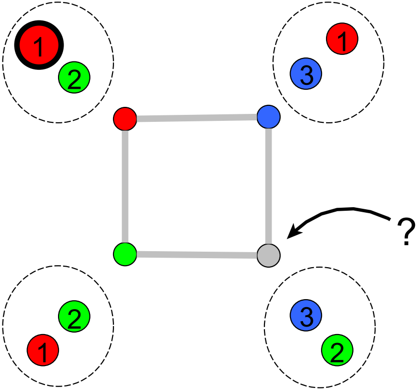


Turns out: C_4 *does* admit a packing for every 3-list assignment.

Thus may make sense to define

Definition

List-packing number $\chi_\ell^\star(G) :=$ smallest k such that G admits an L-packing for every k-list assignment L.

Question: does $\chi_\ell^\star(G)$ exist for every graph?

First attempt: If k much larger than $\chi_\ell(G)$, can greedily find *many* disjoint L-colourings. But then no partition guaranteed!

Answer: yes, $\chi^\star_\ell(G)$ always exists. Follows for instance from:

Theorem (CCDK, 2021+)

For every graph G on n vertices,

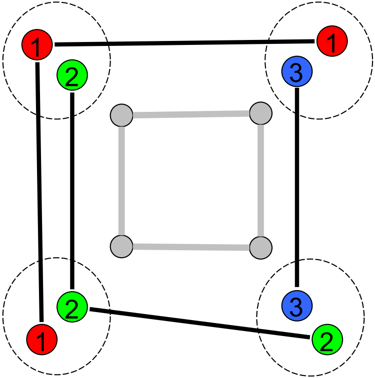
$$\chi^\star_\ell(G) \leq n, \text{ with equality iff } G = K_n.$$

We also study a related stronger parameter $\chi_c^\star(G)$ called the *correspondence packing number*.

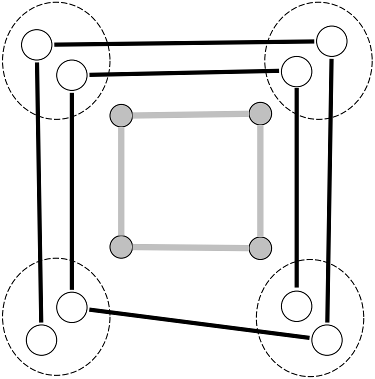
All you need to remember:

$$\chi_\ell^\star(G) \leq \chi_c^\star(G) \text{ for every } G.$$

Correspondence colouring: arbitrary matchings between lists in the cover graph $\mathcal{B}_c(G)$ allowed.



$\mathcal{B}_\ell(G)$ dictated by the colours



$\mathcal{B}_c(G)$ with arbitrary matchings

$\chi_c^\star(G) :=$ smallest k s.t. every size k cover $\mathcal{B}_c(G)$ of G admits a partition into k independent sets of size $|V(G)|$.

All you need to remember:

$$\chi_\ell^\star(G) \le \chi_c^\star(G) \text{ for every } G.$$

Recall:

$$\chi_\ell^\star(G) \le n.$$

Wide open (Catlin, Fischer, Kühn and Osthus, Yuster, 1980–2021):

$$\chi_c^\star(G) \le n + 1?$$

Best known (Yuster, 2021):

$$\chi_c^\star(G) \le (1.78 + o(1))n.$$

Intermezzo: simple bounds for χ_ℓ

Definition

A graph is *d-degenerate* if any subgraph of it contains a vertex of degree at most d. The *degeneracy* $d(G)$ is the smallest d s.t. G is d-degenerate.

A simple bound:

Lemma

For every graph G,

$$\chi_\ell(G) \leq 1 + d(G).$$

Proof: induction on # vertices. Colour vertex of degree of at most $d(G)$.

Some simple bounds:

Lemma

$$\chi_\ell(G) \leq 1 + d(G) =: 1 + \text{ degeneracy of } G.$$

Corollary

$$\chi_\ell(G) \leq 1 + \Delta(G) =: 1 + \text{ maximum degree of } G$$

Corollary

$$\chi_\ell(G) \leq n =: \text{ number of vertices of } G \text{ .}$$

Two influential more involved bounds:

Theorem (Erdős, Rubin, Taylor, 1980)

$\chi_\ell(G) \leq \log_2(n) + 1$ for every bipartite G on n vertices.

Theorem (Johansson, 1996 ; Molloy, 2019)

$$\chi_\ell(G) \leq (1 + o(1))\frac{\Delta(G)}{\log(\Delta(G))} \text{ for every triangle-free } G.$$

Extending bounds to list-packing

Recall:

$$\chi_\ell(G) \le d + 1, \text{ for every } d\text{-degenerate } G.$$

Theorem (CCDK, 2021+)

For any d-degenerate graph G,

$$\chi_\ell^\star(G) \le \chi_c^\star(G) \le 2d.$$

Conversely, for every integer $d \ge 2$, there exists a d-degenerate graph G with $\chi_c^\star(G) = 2d$ and $\chi_\ell^\star(G) \ge d + 2$.

Theorem (Alon, 1993 and 2000)

$$\chi_\ell(G) \geq C \cdot \log(d(G)),\ \text{for some uniform constant } C > 0.$$

Combining with result that $\chi_\ell^\star(G) \leq 2d(G)$ yields:

Corollary

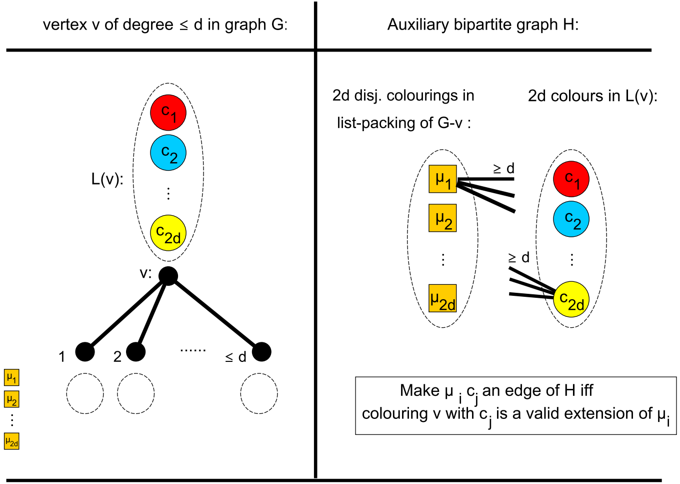
$$\chi_\ell^\star(G) \leq c^{\chi_\ell(G)},\ \text{for some uniform constant } c > 1.$$

Main open question: is there a $c \geq 2$ such that

$$\chi_\ell^\star(G) \leq c \cdot \chi_\ell(G)?$$

Upper bound $\chi_\ell^\star(G) \leq 2 \cdot d$

Upper bound uses inducton on # vertices
and Hall's Marriage Theorem.



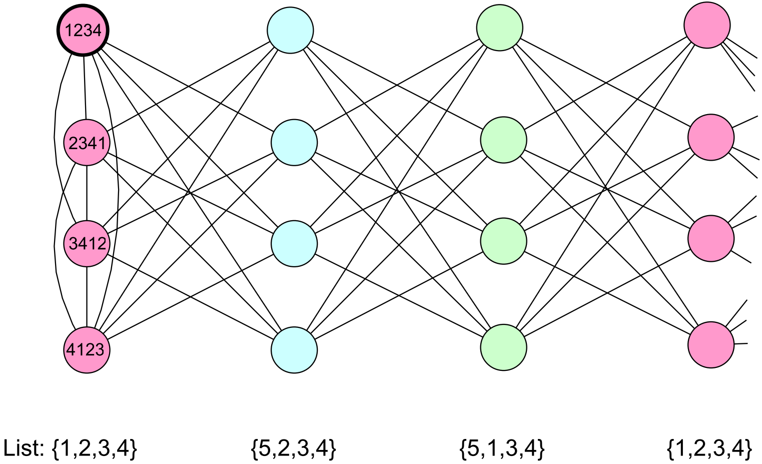
H has minimum degree $\geq d$ and Hall's Marriage Theorem $\Rightarrow$ *H* has a perfect matching $\Rightarrow$ can extend list-packing of $G - v$ to G. □

Lower bound construction: $\chi_\ell^\star(G) \geq d+2$

Example for $d = 3$.

Goal: construct a 3-degenerate graph G with a 4-list assignment L such that G is not L-colourable.

Lower bound construction: $\chi_\ell^\star(G) \geq d+2$



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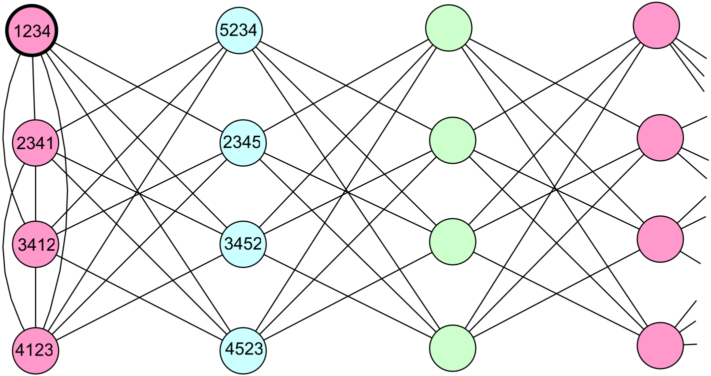
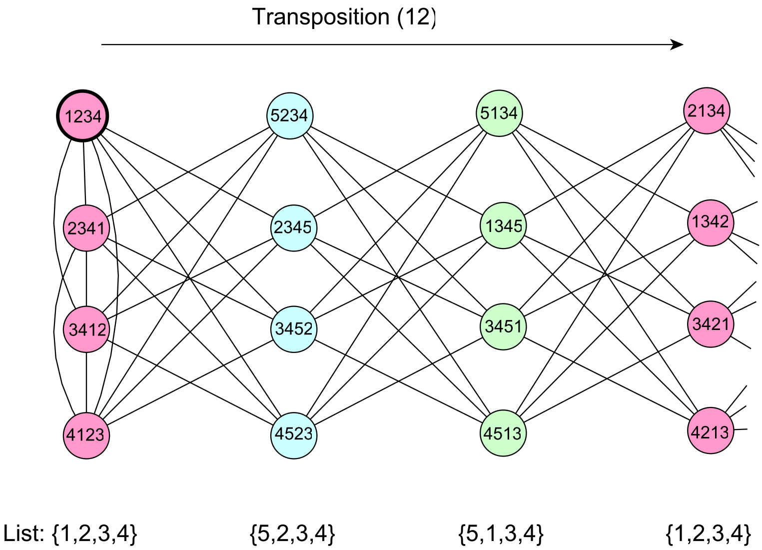


List: {1,2,3,4} {5,2,3,4} {5,1,3,4} {1,2,3,4}

Lower bound construction: $\chi_\ell^\star(G) \geq d + 2$



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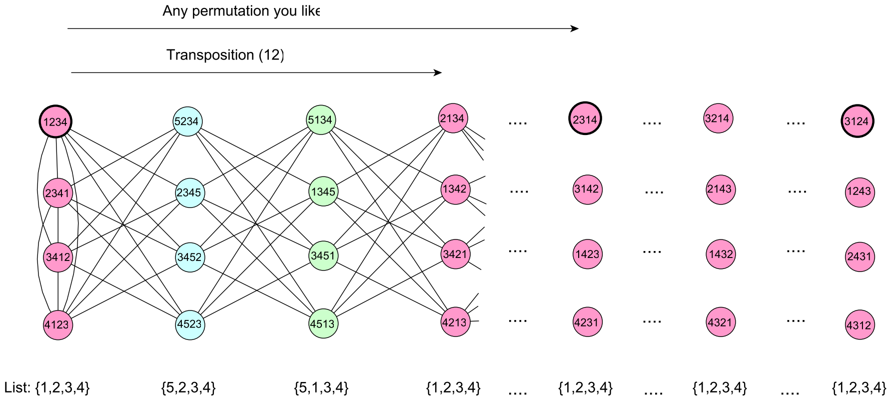

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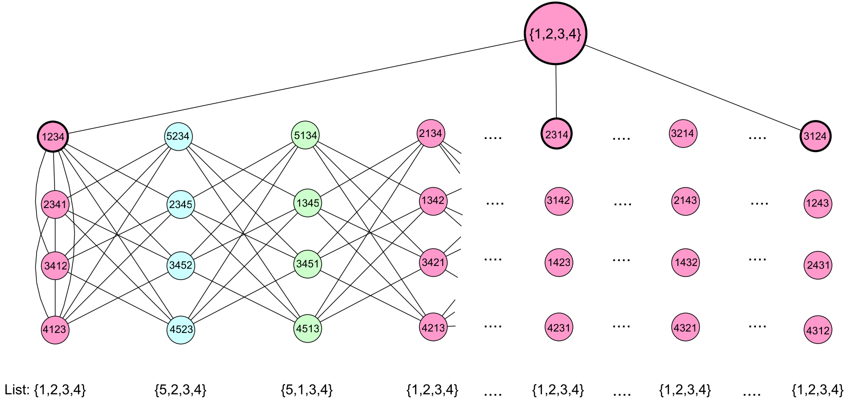


A 3-degenerate graph with an uncolourable 4-list assignment.
So $\chi_\ell^\star \geq 5$.

We have seen that the optimal bounds on χ_ℓ resp. $\chi_\ell^\star$ in terms of *degeneracy* are distinct. What about *maximum degree*?
Recall: $\chi_\ell(G) \leq \Delta(G) + 1$.

Question

$$\text{Is } \chi_\ell^\star(G) \leq \Delta(G) + 1 \text{ ?}$$

Work in progress:

$$\text{Yes if } \Delta(G) \leq 3.$$

Also 'yes' if G bipartite. In general, we only know:

Theorem (CCDK, 2021+)

$$\chi_\ell^\star(G) \leq \Delta(G) + \chi_\ell(G) + 1$$

Bipartite bounded degree

Recall:

$$\chi_\ell(G) \leq (1 + o(1))\frac{\Delta(G)}{\log(\Delta(G))},$$

if G is triangle-free.

Theorem (CCDK, 2021+)

For every bipartite G,

$$\chi_\ell^\star(G) \leq (1 + o(1))\frac{\Delta(G)}{\log(\Delta(G))}.$$

Remark: the same bound holds for the related stronger parameter $\chi_c^\star(G)$; which is sharp up to factor 2!

Bipartite on n vertices

Recall:

$$\chi_\ell(G) \le \log_2(n) + 1$$

if G bipartite on n vertices.

Theorem (CCDK, 2021+)

For graphs G on n vertices, we have as $n \to \infty$,

$$\chi_\ell^\star(G) \le \begin{cases} (1+o(1))\log_2 n & \text{if } G \text{ bipartite,} \\ (1+o(1))\chi(G)\log n & \text{if } \chi(G) \text{ uniformly bounded as } n \to \infty, \\ (5+o(1))\chi(G)\log n & \text{in general.} \end{cases}$$

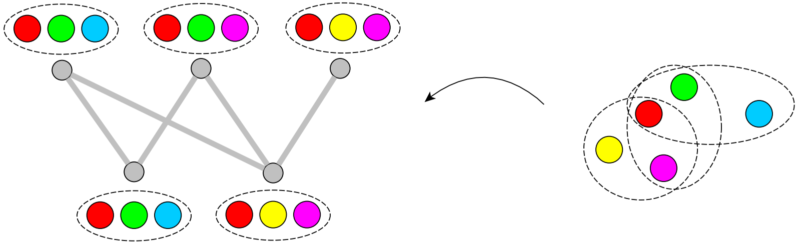
Remark: asymptotically matches the best bounds for $\chi_\ell(G)$.

Bipartite on n vertices

$\chi^\star_\ell(G) \le (1+o(1))\log_2(n)$ if G bipartite on n vertices.

Proof sketch:

Let L be a k-list-assignment of a bipartite graph G.
Consider a uniformly random mapping from the *union* of the lists $\bigcup_v L(v)$ to $\{0,1\}$.



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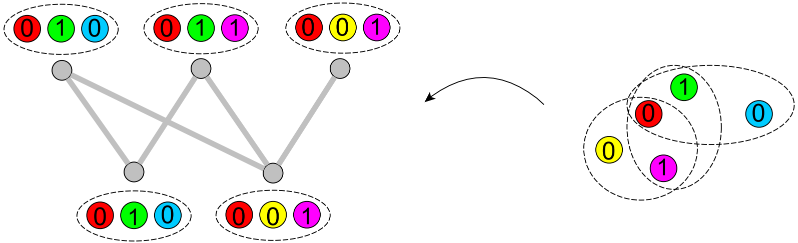

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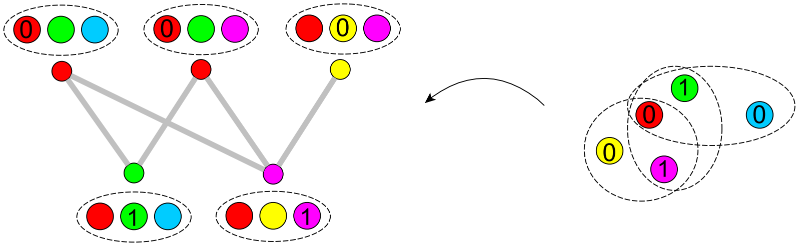
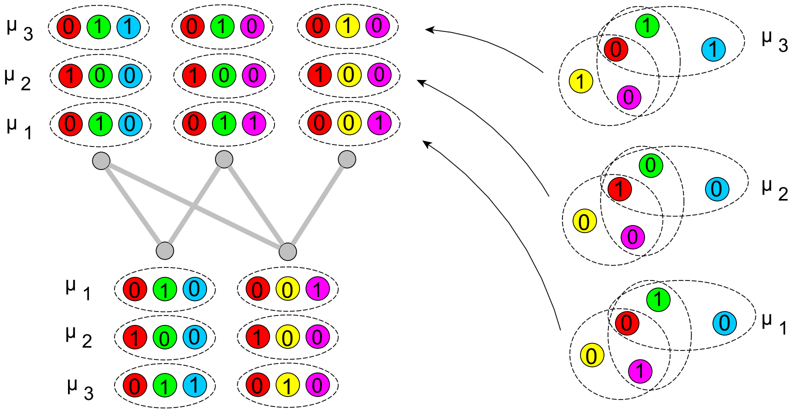


Can extract a single list-colouring if every list at the top contains a 0 and every list at the bottom contains a 1.

Bipartite on n vertices

Now consider k *independent* uniformly random such mappings $\mu_1, \ldots, \mu_k$ from the list-union $\bigcup_v L(v)$ to $\{0, 1\}$.



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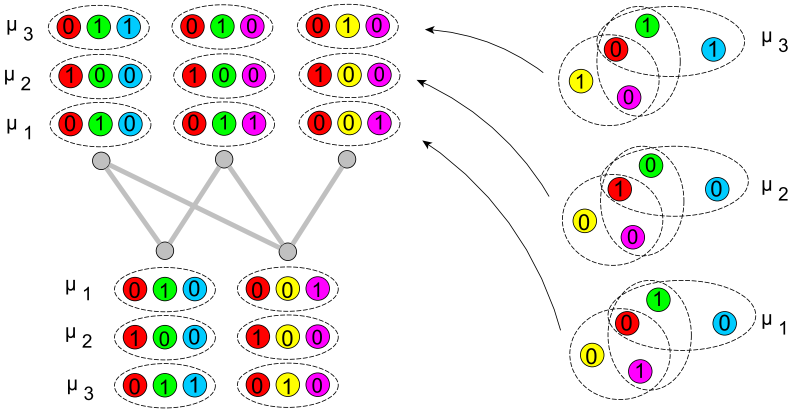

Defines a *random binary* $k \times k$ matrix $M(v)$ for each vertex v.

Bipartite on n vertices

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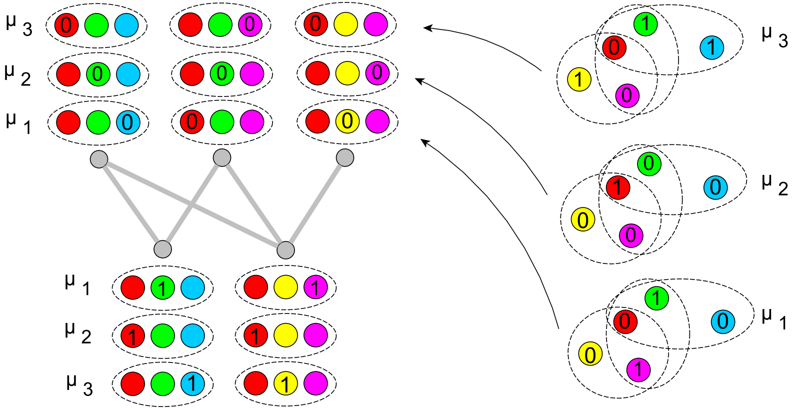

Can extract a list-packing if every matrix at the top contains a 0-*transversal* and every matrix at the bottom contains a 1-*transversal*.

Definition

For a vertex v the *bad event* BAD(v) occurs if its associated random matrix has no 0-transversal (if v is at the top), resp. has no 1-transversal (if v is at the bottom).

We saw: there is a list-packing if non of the bad events BAD(v) occurs.

So **suffices** to show:

$$\text{Prob}(\text{BAD}(v) \text{ occurs for some } v) < 1.$$

We use:

Key Lemma (CCDK 2021+, Everett and Stein 1973)

Let $0 < p \leq 1$. Let A be a random $k \times k$-matrix with independent Bernoulli(p) distributed elements. Then

$$\text{Prob}\,(A \text{ has no 1-transversal}) = 2k(1-p)^k(1+o(1)) \text{ as } k \to \infty.$$

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Implies for every vertex v:

$$\text{Prob}(\text{BAD}(v) \text{ occurs}) \leq (1+o(1)) \cdot 2k \left(\frac{1}{2}\right)^k.$$

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So by union bound,

$$\text{Prob}(\text{BAD}(v) \text{ occurs for some } v) \leq (1+o(1))n \cdot \frac{k}{2^{k-1}} < 1$$

as $n \to \infty$,

provided the lists have size $k \geq (1+o(1))\log_2(n)$.

□

Concludes proof that

$\chi_\ell^\star(G) \leq (1+o(1))\log_2(n)$ if G bipartite on n vertices.

Summary

We studied successively stronger parameters,

$$\chi \le \chi_\ell \le \chi_\ell^\star \le \chi_c^\star.$$

Proved several bounds on $\chi_\ell^\star, \chi_c^\star$ that $\sim$ match the best bounds on χ_ℓ.

Open problems

- Planar graphs? We know $5 \leq \chi_\ell^\star(G) \leq 10$.
- If G is $d-$degenerate? We know $d + 2 \leq \chi_\ell^\star(G) \leq 2d$.
- Maximum degree: is $\chi_\ell^\star(G) \leq \Delta(G) + 1$?
- Colouring the edges instead of the vertices.
- Is $\chi_\ell^\star(G) \leq c \cdot \chi_\ell(G)$, for some constant $c \geq 2$?
- ...

Thank you for your attention!

For each vertex v, let $L(v)_j$ denote the j-th colour in $L(v)$, wrt some arbitrary order. Then form the $k \times k$ matrix M_v given by

$$M_v(i,j) = \mu_i(L(v)_j)$$

M_v has a 1-*transversal* if it has k elements equal to 1 that pairwise do not share a row or column. (similar for 0-transversal)
For $v \in A$, a *bad event* is that M_v does not have a 0-transversal. For $v \in B$, a *bad event* is that M_v does not have a 1-transversal.

If no bad event occurs, we can construct a list-packing $c_1, \dots, c_k$: Choose the guaranteed transversal of M_v and for each $i \in [k]$, choose $c_i(v)$ to be the element of $L(v)$ that corresponds to the i-th element of that transversal.

- Because we choose according to a transversal, $c_1(v), \dots, c_k(v)$ indeed form a partition of $L(v)$.
- Each colouring c_i is proper because on A we only choose colours that have been mapped to "0" by μ_i, while on B we only choose colours that have been mapped to "1 " by μ_i.

Remains to show: there exists no bad event. We derive:

Let $0 < p \leq 1$ be a real number. Let A be a random $k \times k$-matrix with independent Bernoulli(p) distributed elements. Then

A has no $1-$transversal, with probability $2k(1-p)^k(1+o(1))$ as $k \to \infty$.

Implies: Prob(bad event for $M(v)) \leq (1+o(1)) \cdot 2k(\frac{1}{2})^k$.
By union bound,

$$\text{Prob(some bad event occurs)} \leq (1+o(1))n \cdot k/2^{k-1} < 1$$

as $n \to \infty$, provided $k \geq (1+o(1)) \log_2(n)$.

□