Large independent sets in triangle-free subcubic graphs: beyond planarity

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Université Libre de Bruxelles

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G graph on n vertices

Independence number: $\alpha(G) = \max\{k \mid \exists k \text{ pairwise nonadjacent vertices}\}$

'Subcubic' $=$ maximum degree at most 3

How large is $\frac{\alpha(G)}{n}$ in subcubic triangle-free graphs G?

Independence ratio.

In general, determining $\alpha(G)$ is *NP-hard*, even when restricting to planar subcubic triangle-free graphs.

Independence ratio.

In general, determining $\alpha(G)$ is *NP-hard*, even when restricting to planar subcubic triangle-free graphs.

$\Rightarrow$

More feasible goal:

Given a (nice) graph class $\mathcal{G}$, find largest $c \in (0,1]$ such that for all $G \in \mathcal{G}$:

$$\frac{\alpha(G)}{n} \geq c.$$

Planar graphs

Theorem (four-colour theorem, 1976, Appel and Haken)

G planar $\Rightarrow$ G is 4-colourable.

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G planar $\Rightarrow G$ is 4-colourable.

Corollary

G planar $\Rightarrow \alpha(G) \geq \frac{n}{4}$.

Every known proof uses the four-colour theorem!

Planar graphs

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Corollary

G planar $\Rightarrow \alpha(G) \geq \frac{n}{4}$.

What if we also forbid triangles?

Triangle-free planar

Theorem (1959, Grötzsch)

G triangle-free planar $\Rightarrow$ G is 3-colourable $\Rightarrow \alpha(G) \geq \frac{n}{3}$.

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Theorem (1985, Jones)

G triangle-free planar $\Rightarrow \alpha(G) \geq \frac{n+1}{3}$.

Sharp for infinitely many graphs

Graph class	$\alpha \geq$	Due to
planar	$\frac{n}{4}$	4-colour theorem (1976)
planar triangle-free	$\frac{n+1}{3}$	Grötzsch (1959), Jones (1985)

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Can we do better than $\frac{1}{3}$?

Yes, if we assume G is subcubic.

Subcubic triangle-free

Theorem (1979, Statton)

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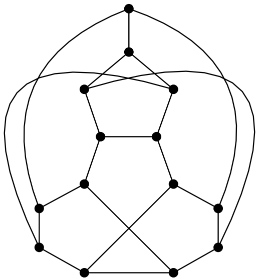
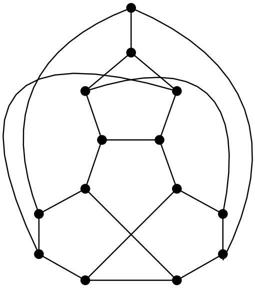
Only two tight examples among connected graphs [Heckman '08].



$n = 14 \quad \alpha = 5$



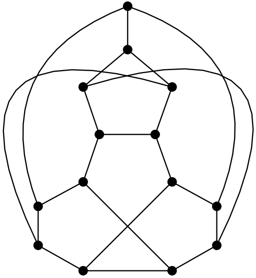
$n = 14 \quad \alpha = 5$

Subcubic triangle-free and connected

Theorem (1995, Fraughnaugh and Locke)

G connected subcubic triangle-free $\Rightarrow \alpha(G) \geq \frac{11}{30}n - \frac{4}{30}$.

The same two graphs are the only tight examples:



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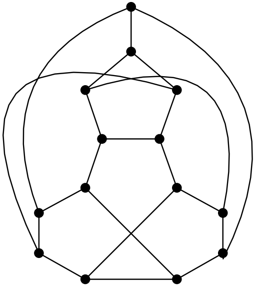

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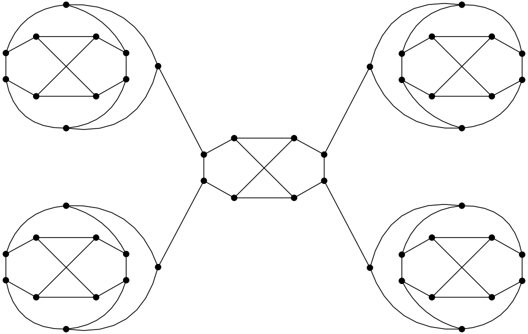

Asymptotically tight; infinitely many graphs with $\alpha = \frac{11}{30}n - \frac{2}{15}$.

Graph class	$\alpha \geq$	Due to
planar	$\frac{n}{4}$	4-colour theorem (1976)
planar triangle-free	$\frac{n+1}{3}$	Grötszch (1959), Jones (1985)
subcubic triangle-free	$\frac{5n}{14}$	Statton (1979)
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Can we do better than $\frac{11}{30}$?

Yes, if we additionally assume G is planar.

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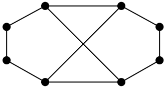
connected subcubic triangle-free	$\frac{11n-4}{30}$	Fraughnaugh and Locke (1995)
planar subcubic triangle-free	$\frac{3n}{8}$	Conjectured (1976) by Albertson, Bollóbas and Tucker, proved by Heckman and Thomas (2006)

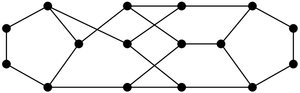
Theorem (2006, Heckman and Thomas)

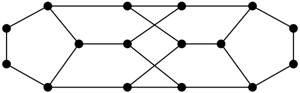
G planar subcubic triangle-free $\Rightarrow \alpha(G) \geq \frac{3}{8}n$

Sharp for infinitely many *bad* graphs, e.g. :



$\alpha = 3, n = 8.$





$\alpha = 6, n = 16.$

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Question: can we relax the planarity condition?

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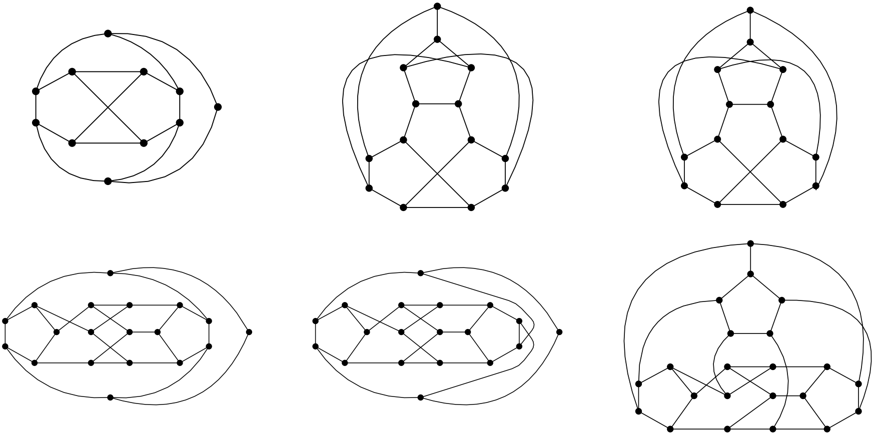
Old conjecture: suffices to forbid six non-planar subgraphs

Relaxing planarity

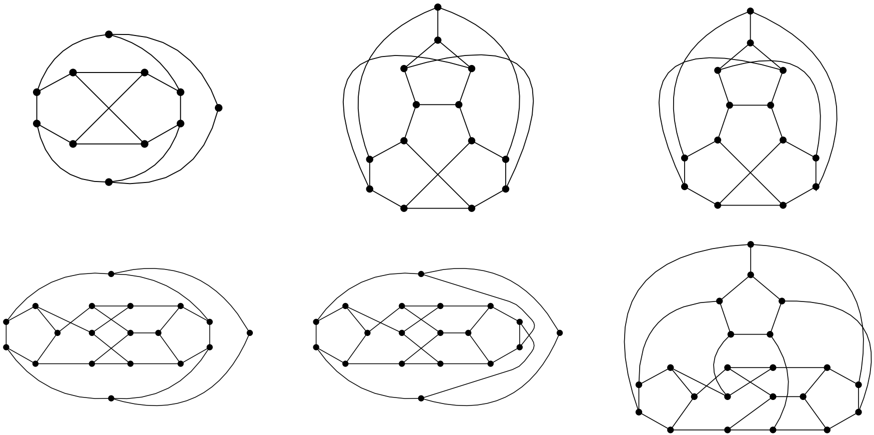
Instead of planarity:
family $\mathcal{F}$ of six non-planar forbidden induced subgraphs?



Relaxing planarity

Conjecture (1995, Fraughnaugh and Locke + Bajnok and Brinkmann)

G $\mathcal{F}$-free subcubic triangle-free $\Rightarrow \alpha(G) \geq \frac{3}{8}n$.



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Definition: a graph G (on at least three vertices) is called 2-connected if $G - v$ is connected for every vertex v.

Relaxing planarity

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Theorem (2019+, C., Goedgebeur and Joret)
Both conjectures are true :-)

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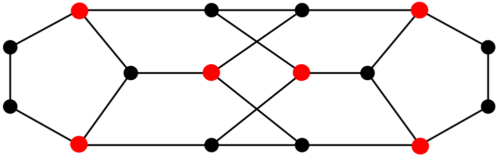
Our technical theorem generalizes all of the above results concerning subcubic graphs

Graph class	$\alpha \geq$	Sharp due to
planar	$\frac{n}{4}$	∞ many
planar triangle-free	$\frac{n+1}{3}$	∞ many
subcubic triangle-free	$\frac{5n}{14}$	∞ many (ess. two members of $\mathcal{F}$)
connected subcubic triangle-free	$\frac{11n-4}{30}$	two members of $\mathcal{F}$
planar subcubic triangle-free	$\frac{3n}{8}$	∞ many
$\mathcal{F}$-free subcubic triangle-free	$\frac{3n}{8}$	∞ many graphs that are 3-connected, girth 5, non-planar and in fact with arbitrarily large clique minors
2-connected subcubic triangle-free	$\frac{3n-2}{8}$	three members of $\mathcal{F}$

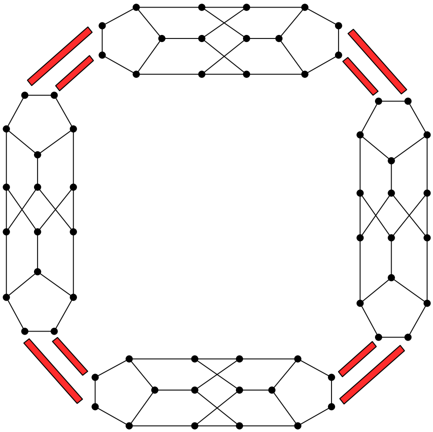
∞ many sharp examples



$$\alpha = 6 = \frac{3n}{8}.$$

Use as a building block.

∞ many sharp examples



3-connected, non-planar and $\alpha = \frac{3n}{8}$.

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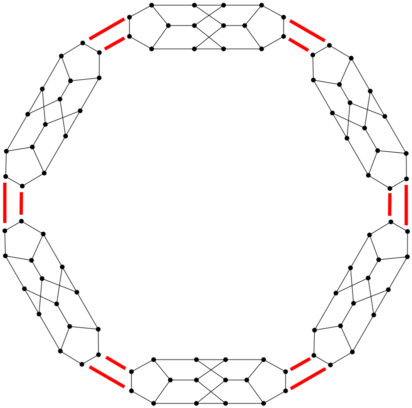

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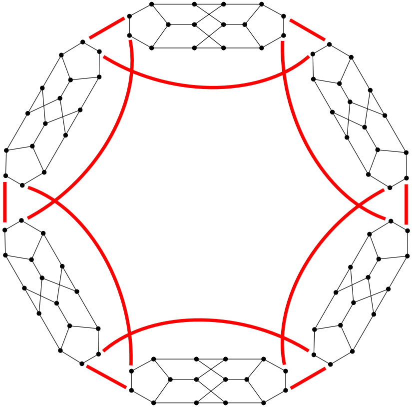

3-connected, girth 5, non-planar and $\alpha = \frac{3n}{8}$.

Motivation for the technical theorem

Theorem, C., Goedgebeur, Joret (2019+)

If G is $\mathcal{F}$-free subcubic triangle-free, then $\alpha(G) \geq \frac{3n}{8}$.

Enough to show the statement when

- G connected, and
- G *critical*, meaning $\alpha(G - e) > \alpha(G) \quad \forall e \in E(G)$

A sparsity measure:

$$\mu := \frac{9n_3 + 10n_2 + 11n_1 + 12n_0}{24} = \frac{6n - |E(G)|}{12}$$

where $n_i :=$ number of vertices of degree i.

Remarks:

$\mu \geq \frac{3}{8}n$;

$\left\lceil \frac{3}{8}n - \frac{1}{12} \right\rceil \geq \frac{3}{8}n$ because $n \in \mathbb{Z}$.

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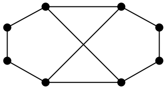
Hence, to show $\alpha \geq \frac{3}{8}n$ it is enough to prove $\boxed{\alpha \geq \mu - \frac{1}{12}}$

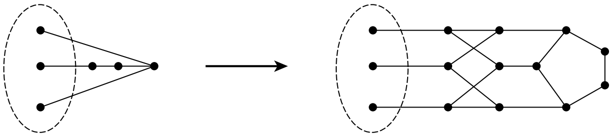
Problem: there exist bad graphs with $\alpha = \mu - \frac{2}{12}$.

Definition of bad graphs



is bad and

every 8-augmentation of a bad graph is bad:



Attempt 2: Show that $\alpha \geq \mu - \frac{1}{12}$, unless G is bad.

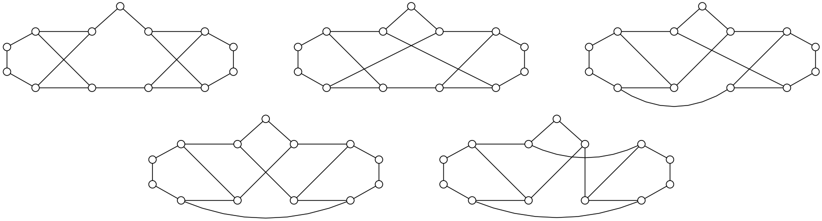
This is true, however ...

Problem: to prove it, we need to consider a slightly stronger statement. This involves dangerous graphs, some of which attain $\alpha = \mu - \frac{1}{12}$.

Dangerous graphs

Technical recursive definition.

(C_5 and the following are the smallest dangerous graphs)



Main technical theorem (CvB-G-J '19+)

Suppose

- G $\mathcal{F}$-free subcubic and triangle-free
- G connected and critical, and
- G not bad

then $\alpha \geq \mu - \frac{1}{12}$.

Main technical theorem (CvB-G-J '19+)
Suppose

- G $\mathcal{F}$-free subcubic and triangle-free
- G connected and critical, and
- G not bad

then $\alpha \geq \mu - \frac{1}{12}$.

If moreover

- G has ≥ 3 degree-2 vertices and
- G not dangerous and has no bad subgraph

then $\alpha \geq \mu$.

Generalization: allowing triangles

Define the refined measure

$$\mu^*(G) := \frac{6n - e - 2t}{12},$$

where e is the number of edges in G and t is the maximum number of vertex-disjoint triangles.

Theorem (2019+, C, Goedgebeur and Joret)

Let G be a critical connected subcubic graph which is not isomorphic to K_4 or any member of $\mathcal{F}$. Then

- $\alpha(G) = \mu^*(G) - \frac{2}{12}$ if G is bad or almost bad
- $\alpha(G) \geq \mu^*(G) - \frac{1}{12}$ otherwise.

Plan of the proof

G minimum counter-example

- G almost 3-connected: If X is a 2-cutset then $G - X$ has exactly two components, with one isomorphic to K_1 or K_2
- G has no bad subgraph
- Deal with degree-2 vertices:
    - case where neighbors have both degree 2
    - case where neighbors have both degree 3
    - case where neighbors have degree 2 and 3

$\rightarrow$ G is cubic and 3-connected

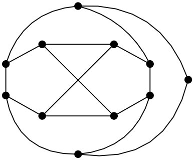
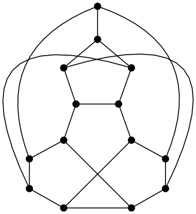
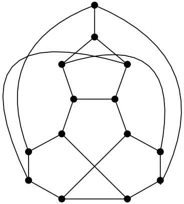
- G has no 4-cycle
- G has no 6-cycle
- G has no dangerous subgraph (in particular, no 5-cycle)

Final argument: Local structure around a shortest even cycle

Open problems

G $\mathcal{F}$-free subcubic triangle-free

- $\alpha(G) \geq \frac{3n}{8}$ is still best possible if we also forbid four-cycles. What happens for even larger girth?
- Fractional chromatic number $\chi_f(G)$ at most $\frac{8}{3}$? (cp. conjecture Heckman and Thomas '08)

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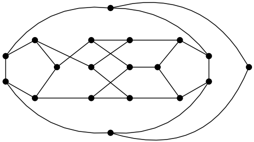
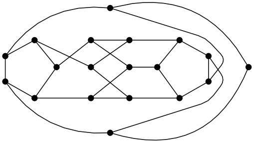
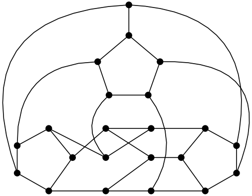




An empty slide

Open problems

Staton '79 If G subcubic and triangle-free then $\frac{n}{\alpha} \leq \frac{14}{5}$

Recall: $\frac{n}{\alpha} \leq \chi_f$

Dvořák, Sereni, Volec '14 (conjectured by Heckman & Thomas '01)
If G subcubic and triangle-free then $\chi_f \leq \frac{14}{5}$

Could the upper bound on χ_f be improved if we further assume

- G connected, or
- G 2-connected, or
- G planar, or
- G has none of the 6 exceptional graphs as subgraph?

Plan of the proof; how the graphs of $\mathcal{F}$ emerge.

G minimum counter-example

- G almost 3-connected: If X is a 2-cutset then $G - X$ has exactly two components, with one isomorphic to K_1 or K_2.
- G has no bad subgraph. $\Rightarrow F_{11}, F_{19}^{(1)}, F_{19}^{(2)}$.
- Deal with degree-2 vertices:
  - case where neighbors have both degree 2.
  - case where neighbors have both degree 3 $\Rightarrow$ red. to dangerous graphs.
  - case where neighbors have degree 2 and 3 $\Rightarrow$ red. to 8-augmentation.

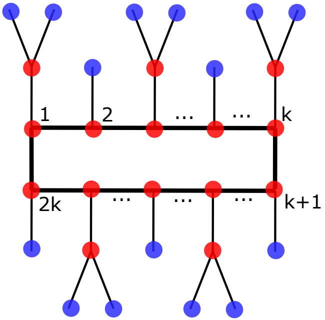
$\rightarrow$ G is cubic and 3-connected $\Rightarrow$ henceforward only need: $\alpha \geq \mu - \frac{1}{12}$.

- G has no 4-cycle
- G has no 6-cycle $\Rightarrow F_{14}^{(1)}, F_{14}^{(2)}$.
- G has no dangerous subgraph (in particular, no 5-cycle) $\Rightarrow F_{22}$.

Final argument: Local structure around a shortest even cycle

Final argument (simplified):

Shortest even cycle $1, 2, \ldots, 2k$ in G.
$G' := G$ minus red vertices.



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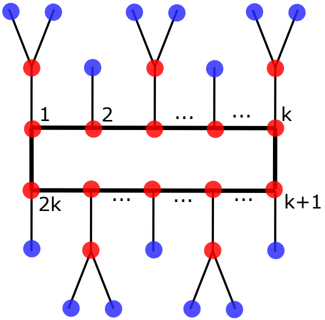

$$\alpha(G) - \alpha(G') \geq k = 3k \cdot \frac{9}{24} - k \cdot \frac{1}{24} = \mu(G) - \mu(G').$$

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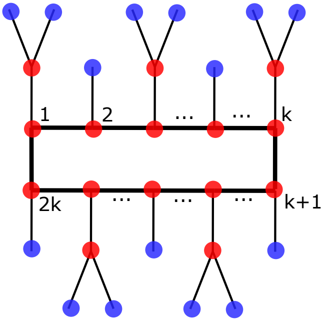
Shortest even cycle $1, 2, \ldots, 2k$ in G.
$G' := G$ minus red vertices.



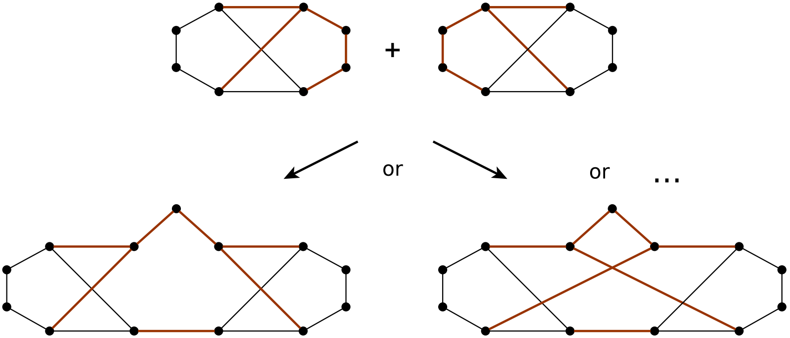
$$\begin{aligned} \alpha(G) - \alpha(G') &\geq k \\ &= 3k \cdot \frac{9}{24} - k \cdot \frac{1}{24} = \mu(G) - \mu(G'). \end{aligned}$$

Apply induction to $\alpha(G') - \mu(G')$ for desired bound on $\alpha(G) - \mu(G)$. □

Definition dangerous graphs

C_5 is dangerous

Join of two bad graphs is dangerous:



$\alpha = \mu - \frac{1}{12}$ if G dangerous.