Cliques, colours and clusters

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2018, Dissertation

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Proefschrift

ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus prof. dr. J.H.J.M. van Krieken,
volgens besluit van het college van decanen
in het openbaar te verdedigen op woensdag 27 juni 2018
om 14:30 uur precies

door

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geboren op 10 december 1988
te Den Haag

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The research in this thesis has been financially supported by the Netherlands Organisation for Scientific Research (NWO) under grant no. 613.001.217, as well as a bilateral project PHC Van Gogh 2016 (35513NM).

Voor papa en mama

Contents

Chapter 1

Introduction

In this thesis, we discuss problems concerning extremal graph theory, colouring of graphs and percolation theory.

Each chapter is based on a paper and as such can be read independently. A common feature among these papers is that we investigate the size and properties of objects that are in some sense highly connected (*cliques* of a graph, connected *clusters* in a random percolating subgraph) or not connected at all (stable sets of a graph, each of which we want to assign a *colour*). Hence the title of this thesis: cliques, colours, clusters.

In this introduction we provide general background and then highlight some aspects of each chapter. We mostly focus on the chromatic number of graphs. Please note the dividers placed carefully throughout this introduction, marking the transition to a new subject. For a more complete image of the obtained results, we refer the reader to the abstract and introduction at the start of each chapter.

Clique, stable set, chromatic number

Let G be a (simple, loopless) graph with vertices $V(G)$ and edges $E(G)$. In this thesis, we are often interested in the *chromatic number* $\chi(G)$ of a graph G. By definition, $\chi(G)$ is the least number of colours one needs to colour the vertices of $V(G)$, such that every two adjacent vertices have different colours. A vertex-colouring that respects this rule is called a *proper colouring*.

To better understand the chromatic number, it is convenient to introduce cliques and stable sets. A *clique* of G is a subset $C \subseteq V(G)$ such that there is an edge between any two vertices of C. The *clique number* $\omega(G)$ of G is the size of a largest clique in G. If all n vertices of a graph form a clique then we call the graph *complete* and we denote it by K_n. On the other hand, a *stable set* of G is a subset $S \subseteq V(G)$ such that there is *no* edge between any pair of vertices in S. The *stability number* $\alpha(G)$ of a graph G is the size of the largest stable set in G.

Since all vertices in a stable set can be assigned the same colour, $\chi(G)$ can alternatively be defined as the least number of stable sets that partition $V(G)$. From this alternative definition, it immediately follows that $\chi(G) \geq |V(G)|/\alpha(G)$.

It is an important and natural task to find optimal upper bounds on the chromatic number for basic classes of graphs. A famous example is the Four Colour Theorem [53, 54, 93] for *planar graphs*. A planar graph is a graph that can be embedded in the plane such that its edges only intersect at their endpoints. In the language of graphs, the Four Colour Theorem states that any planar graph has chromatic number at most 4. A popular interpretation is that for any hypothetical world map, at most four colours are required to colour the countries of the map so that no two bordering countries have the same colour. This fact has the following crude application in wireless networks with a planar geometry. To avoid interference between neighbouring transmitters, it is desirable to have them transmit their signals in different frequency ranges. According to the Four Colour Theorem, only four different frequency ranges are required to avoid interference, thus enabling efficient use of bandwidth. Other applications of chromatic graph theory are in optimally allocating or scheduling tasks. One can construct a graph such that each task is represented by a vertex and any two conflicting tasks (that cannot be executed at the same time or allocated to the same location because e.g. they require the same resource) are connected by an edge. An optimal colouring then yields an allocation that minimizes the total required time or space.

In practice, one would often need more specialized variants of the concept of chromatic number. Approximations that almost always work rather than exact results would be deemed sufficient and the focus would be on algorithms rather than on existence results. Although our results have some bearing on such aspects, instead our immediate interest lies in more theoretical questions: existence and (close to) sharp bounds for worst cases.

Upper bounds for chromatic number

Determining whether a given graph is $k-$colourable is computationally hard (NP-complete) for any $k \geq 3$ and despite many decades of research, there are not many general purpose techniques available to find good upper bounds. We will now discuss a few of such techniques.

Let us first have a look at 'greedy' colouring. Let $\Delta(G) = \max_{v \in V(G)} \deg(v)$ denote the *maximum degree* of G. Then it holds that $\chi(G) \leq \Delta(G) + 1$. Indeed, take a vertex $v \in V(G)$ and remove it from G. By induction we may assume that $G - v$ is $(\Delta(G)+1)$-colourable. Since v has at most $\Delta(G)$ neighbours, we can colour v with a colour not appearing among its neighbours, thus obtaining a proper $(\Delta(G) + 1)-$colouring of G. This greedy procedure can be refined with the concept of degeneracy. A graph G is called $k-$*degenerate* if every induced subgraph of G has a vertex of degree k or less. The *degeneracy* $\delta^*(G)$ of G is the least k such that G is k-degenerate. Exactly the same induction argument then yields that $\chi(G) \leq \delta^*(G) + 1$. In this spirit, if one wants to show that $\chi(G) \leq b + 1$ for all graphs G in some graph class that is invariant under vertex deletion, one can consider a vertex-minimal counterexample G and (using the special properties of the class) derive the existence of a vertex of degree b in G, leading to a $(b+1)-$colouring of G and thus a contradiction. This type of reasoning is referred to as a degeneracy argument.

In some lucky cases where the structure of the graph is known in detail, it is possible to describe a nice partition of the vertices $V(G) = \bigcup_i V_i$ such that the induced subgraphs $G[V_i]$ have small chromatic number. In that case one colours each of these graphs with their own colour palette, yielding $\chi(G) \leq \sum_i \chi(G[V_i])$. For an example, one could try to partition G into a small number of vertex sets that induce planar graphs and then repeatedly apply the four colour theorem.

A technique that is especially effective for planar graphs is the *discharging method.* Every *plane graph* (which is an embedding of a planar graph in the plane) satisfies Euler's formula $e - v - f = -2$, where e, v and f are the number of edges, vertices and faces of the graph respectively. Suppose one wants to prove that a certain subclass of planar graphs satisfies a statement P. For a contradiction one then assumes the class contains a counterexample G to P. Next, certain suitable *charges* (elements of $\mathbb{R}$) are assigned to the vertices, edges and faces of G, chosen in such a way that the sum of the charges equals $e - v - f$ and thus is negative. Then, using that P is false for G, one proves that it is possible to (locally) redistribute the charges in such a way that ultimately all charges are nonnegative, while also preserving the (negative) sum of the charges; contradiction. To prove the main result in chapter 5 we combine, in particular, a degeneracy argument and the discharging method.

All of the above methods predominantly rely on local properties and modifications of the graph. However, sometimes the chromatic number is determined by the global structure of the graph[1]. In such a situation random colouring techniques may form the best option of attack. A nice example is a recent new proof by Molloy [85] of a result of Johansson [64], stating that the chromatic numer of triangle-free graphs is upper bounded by $O\left(\Delta(G)/\ln(\Delta(G))\right)$. Johansson used an iterative scheme to randomly colour subsets of the uncoloured vertices, using concentration bounds in each step to ensure that no two adjacent vertices receive the same colour, with sufficiently high probability. In his new proof, Molloy takes a partial proper colouring uniformly at random among all partial proper colourings [2] that use at most $(1 + o(1))\Delta(G)/\ln(\Delta(G))$ colours. Here 'partial' means that not all vertices need to have a colour assigned to them. Those uncoloured vertices are called blank. Subsequently he shows that with nonzero probability the neighbourhood of each blank vertex has been coloured in such a way that the blank vertices can be coloured greedily, with a degeneracy argument. One of the nice aspects of the proof is that the uniformly random partial colouring captures global structure of the graph, yet this random colouring only has to be analyzed locally, thus making the analysis feasible.

Lower bounds for the chromatic number

Having discussed techniques for obtaining upper bounds, let us now have a look at lower bounds. Typically one is not interested in a single graph but in a whole class of graphs $\mathcal{G}$ and the largest possible chromatic number occuring in that class, $\max_{G \in \mathcal{G}} \chi(G)$. In that case we only need to find *one* graph in $\mathcal{G}$ which has large chromatic number. Two examples for $\mathcal{G}$ are the class of planar graphs (K_4 is planar and has chromatic number 4, meeting the bound of the four colour theorem) or the class of $H-$free graphs (graphs not containing the graph H as a subgraph) of maximum degree Δ. As mentioned before, all graphs G satisfy $\chi(G) \geq |V(G)|/\alpha(G)$. Therefore, given a class $\mathcal{G}$, it can be fruitful to search $\mathcal{G}$ for a graph of small stability number. For this, the *probabilistic method* often works: one takes a random graph from $\mathcal{G}$ (according to some convenient

[1]To see that a colouring problem can be of a non-local nature, consider the following result of Erdős [38]. For all k there exists $\epsilon > 0$ so that for all sufficiently large n there exist graphs G on n vertices with $\chi(G) > k$ and yet $\chi(G[S]) \leq 3$ for every set S of vertices of size at most ϵn.

[2]Actually a partial proper *list colouring* is used. Given lists $(L(v))_{v \in V(G)}$ of colours, a $L-$list colouring is a proper colouring such that every $v \in V(G)$ receives a colour from its list $L(v)$.

probability distribution) and computes an $A > 0$ (depending on parameters relevant to the class, like the maximum degree or the number of vertices) such that the expected value $\mathbb{E}($ number of stable sets of size $> A$ $)$ is strictly less than 1. It then follows that $\mathcal{G}$ contains at least one graph G without stable sets of size $> A$, implying that $\alpha(G) \leq A$ and so $\chi(G) \geq |V(G)|/A$. This method is often effective because computing expected values is straightforward, even in probability distributions with many dependencies.

Since every two vertices of a clique need to have different colours, $\omega(G)$ is another straightforward lower bound for the chromatic number. In general however, the ratio of $\chi(G)$ and $\omega(G)$ can be arbitrarily large. For example, there exist triangle-free graphs (graphs that do not contain K_3 as a subgraph and thus have clique number at most 2) with arbitrarily large chromatic number. Indeed, there exist explicit constructions of such triangle-free graphs, using the so-called *Mycielskian* [84]. Alternatively, one can also consider the *Erdős-Renyi* graph, which is a random graph on n vertices where each edge independently is chosen to be in the graph with a uniform probability $p \in [0, 1]$. For an appropriate value of $p = p(n)$, one can show that after removing a vertex from each triangle of the Erdős-Renyi graph, there is a positive probability that the chromatic number of the resulting random triangle-free graph is at least $f(n)$, where $f(n)$ is some function tending to infinity as n tends to infinity.

Chi-boundedness

Although the chromatic number cannot be upper bounded by the clique number in general, one may wonder whether it sometimes carries sufficient information for a coarser upper bound. A class of graphs $\mathcal{G}$ is called $\chi-$*bounded* if there exists a *binding function* $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{G}$ and all induced subgraphs H of G, $\chi(H) \leq f(\omega(H))$. Bipartite and planar graphs and graphs of bounded degeneracy are examples of $\chi-$bounded classes. As we have observed, the class of triangle-free graphs is not $\chi-$bounded. For a non-trivial example of a class that is $\chi-$bounded, consider the class of graphs that do not contain any cycle of odd size ≥ 5 as an induced subgraph [96]. *Perfect graphs* have been one of the inspirations for the concept of $\chi-$boundedness. Perfect graphs are graphs for which the chromatic number of every induced subgraph equals the clique number of that subgraph. Thus they have the identity as binding function. The perfect graphs were defined by Claude Berge in the 1960s. They are important objects for graph theory, linear programming and combinatorial optimization. As conjectured by Berge and proved by Lovász, the complement of every perfect graphs is also perfect and this is known as the (weak) perfect graph theorem [11]. Berge also observed that a perfect graph cannot have any odd cycle of size ≥ 5 as induced subgraph (since such cycles have clique number 2 and chromatic number 3), nor can the complement of a perfect graph. Thereupon he conjectured that this property exactly characterizes perfect graphs. This long sought-after important result, known as the strong perfect graph theorem, was finally proved in the 2000s by Chudnovsky, Robertson, Seymour and Thomas [26]. Since its forbidden graph characterization is invariant under taking complements, the strong perfect graph theorem directly implies the weak perfect graph theorem.

Reed's conjecture and fractional chromatic number

We have seen that $\Delta(G) + 1$ and $\omega(G)$ are straightforward upper respectively lower bounds for $\chi(G)$. Reed conjectured [91] that the chromatic number is in fact at most the average of the two, rounded upwards: $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$. This conjecture has been confirmed for several graph classes, among which are the *claw-free graphs* (we will discuss these graphs later) [72]. In general even the case $\omega(G) = 2$ is open, though only for small values of $\Delta(G)$ since triangle-free graphs satisfy $\chi(G) = O(\Delta(G)/\log(\Delta(G)))$.

The *fractional chromatic number* $\chi_f(G)$ of a graph G is a relaxation of the chromatic number. There are multiple equivalent definitions, one of which is the following: $\chi_F(G) \leq k$ iff there exists an integer N and a collection of $k \cdot N$ stable sets of G such that each vertex is contained in exactly N stable sets. Note that this reduces to the definition of $\chi(G)$ if we additionally demand that $N = 1$. It readily follows that $\omega(G) \leq \chi_f(G) \leq \chi(G)$. Furthermore, it can be shown that $\chi_f(G)$ is a rational number for all graphs G, so the fractional chromatic number is truly fractional. For more details on fractional colouring we refer the reader to [95].

Molloy and Reed [87] have proved a fractional analogue of Reed's conjecture. They showed that $\chi_f(G) \leq \frac{\omega(G)+\Delta(G)+1}{2}$ for every graph G. McDiarmid obtained a local refinement (cf [70]): $\chi_f(G) \leq \max_{v \in V(G)} \left(\frac{\deg(v)+\omega(v)+1}{2} \right)$, where $\omega(v)$ denotes the size of the largest clique containing v. Although we do not directly study fractional colourings or Reed's conjecture in this thesis, they did serve as an inspiration in the background, providing hints on what bounds to expect in e.g. chapter 4.

Beyond clique number. Hadwiger number, immersion number.
We have seen that the clique number, being a very local property, is not always sufficient to describe an upper bound for the chromatic number. There are however clique-like structures of a more global nature that do yield such an upper bound. A graph H is a *minor* of another graph G if it can be obtained from G by contracting some edges, deleting some edges and deleting some vertices. Several important graph classes can be characterized by a (finite) set of forbidden minors. For example, the planar graphs are exactly those graphs without K_5 or the complete bipartite graph $K_{3,3}$ as a minor. This is known as Wagner's Theorem. Hadwiger [51] conjectured that for any graph G the chromatic number $\chi(G)$ is at most the size of the largest clique that is a minor of G. This conjectured upper bound is called the *Hadwiger number* of G and denoted by $h(G)$. The conjecture is only proven to hold for $h(G) \leq 5$. Note that by Wagner's Theorem, the four colour theorem is a special case of Hadwiger's conjecture. Regardless of the validity of Hadwiger's conjecture, we know for sure that $\chi(G)$ can be upper bounded by a function of $h(G)$: Kostochka [74] has shown that every graph G is $O(h(G)\sqrt{\log(h(G))})-$degenerate, so that $\chi(G) = O(h(G)\sqrt{\log(h(G))})$.
Another clique number-like global parameter is the *immersion number* $i(G)$ of G. It is the size of the largest clique that is an *immersion* of G, meaning that there is an injective function from the vertices of the clique to $V(G)$ such that the images of the vertices are connected in G by edge-disjoint paths. In spirit of Hadwiger's conjecture it is believed that $\chi(G) \leq i(G)$ for all graphs [81, 1]. The best known result is $\chi(G) < 3.54 \cdot i(G) + 4$ [49]. Also, using a minimal counterexample approach the author of this thesis has derived that $\chi(G) \leq \left\lceil \frac{i(G)+\Delta(G)}{2} \right\rceil$ for all graphs G (see the appendix). Note that this demonstrates a weak variant of Reed's conjecture.

Turán numbers

Several times in this thesis, in chapters 2 and 4 and in some sense also in chapter 3, we investigate a problem under the additional condition that the graphs under consideration do not contain a certain graph H as a subgraph. For H we usually take P_k (a path of order k), C_k (a cycle of order k) or $K_{2,k}$ (the complete bipartite graph on parts of sizes 2 and k). To get a rough idea of the effect of such exclusions on the structure of a graph we here briefly discuss Turán numbers. Given a graph H and an integer n, the Turán number $ex(H,n)$ denotes the maximum possible number of edges in a graph on n vertices that does not have H as a subgraph. Thus the smaller the Turán number, the sparser the graphs must be. For comparison, note that every graph on n vertices has at most $\binom{n}{2} \approx n^2/2$ edges. It turns out that excluding odd cycles does not have a large effect, since bipartite graphs do not contain any odd cycles and yet $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ has $\lfloor n^2/4 \rfloor$ edges. Excluding $K_{2,k}$, an even cycle or a path has a progressively larger effect. For integers k, the following bounds are sharp or close to sharp: $ex(C_{2k+1}, n) = \lfloor n^2/4 \rfloor$ for n sufficiently large, $ex(K_{2,k}, n) = \frac{1}{2}\sqrt{k} n^{3/2} + O(n^{4/3})$, $ex(C_{2k}, n) \leq (k-1)n^{1+1/k} + 16(k-1)n$ and $ex(P_k, n) = \frac{k-2}{2}n$. In particular: the longer the excluded even cycle, the sparser the graph must be. For general non-bipartite H, there exists a good approximation in terms of the chromatic number of H; the classic Erdős-Stone Theorem yields $ex(H,n) \leq ex(K_{\chi(H)}, n) + o(n^2) = (1 - \frac{1}{\chi(H)-1} + o(1)) \cdot \binom{n}{2}$. Even though Turán numbers go all the way back to the early days of extremal graph theory and combinatorics, many problems in this area remain open (cf. e.g.[48]).

Line graph, multigraph, claw-free graph

A proper colouring as discussed up to now only requires that adjacent vertices are coloured differently, but one may desire more strict conditions. We will now discuss a range of graph classes that arise from imposing a different (e.g. distance) requirement on equicoloured parts of the graph.

Let us first consider a colouring of the *edges* such that incident edges are coloured differently. This problem can be described in terms of the chromatic number of *line graphs*. By definition, the line graph $L(G)$ of a graph G has vertices given by the edges of G and, furthermore, two vertices in $L(G)$ are adjacent iff the corresponding edges in G are incident. The chromatic number of the line graph $\chi(L(G))$ is known as the *chromatic index* of G. Note that it equals the minimum number of colours one needs to colour the *edges* of G such that no two incident edges have the same colour. It also equals the minimum number of matchings that partition the edges of G.

Since the edges incident to a vertex of degree $\Delta(G)$ need to have pairwise different colours we have $\chi(L(G)) \geq \Delta(G)$. On the other hand, since each edge is incident to at most $2\Delta(G) - 2$ other edges, a naive upper bound is $\chi(L(G)) \leq \Delta(L(G)) + 1 \leq 2\Delta(G) - 1$, leaving quite some range for the actual value. However, according to a classic theorem of Vizing [102], the upper bound can be improved considerably and in fact the chromatic index is limited to only two possible values: $\chi(L(G)) \in \{\Delta(G), \Delta(G)+1\}$. Having $\chi-$boundedness in the back of our mind, it is worth noting here that $\omega(L(G)) = \Delta(G)$ (provided $\Delta(G) > 2$). For some natural classes it is possible to determine the chromatic index exactly. For example, due to König's Line Colouring Theorem [73], every bipartite graph G has chromatic index exactly $\Delta(G)$.

Vizing's theorem can be generalized to multigraphs. A *multigraph* is a graph which is permitted to have multiple edges between two fixed vertices. The number of such edges connecting a vertex pair is the *multiplicity* of that vertex pair, and the maximum over all vertex pairs is called the multiplicity $\mu(G)$ of the graph G. Vizing's theorem for multigraphs states that $\chi(L(G) \leq \Delta(G) + \mu(G)$. From this, one can derive Shannon's Theorem [98]: all multigraphs G satisfy $\chi(L(G)) \leq \frac{3}{2}\Delta(G)$. This bound is sharp because it is attained by three vertices with $\Delta(G)/2$ edges between each pair of vertices.

Note that each colour class of an edge-colouring of a multigraph H is a matching and therefore contains at most $\lfloor |V(H)|/2 \rfloor$ edges. Therefore all multigraphs G satisfy $\chi(L(G)) \geq D(G)$, where $D(G) := \max_{\substack{H \subseteq G \text{s.t.} \\ |V(H)| \geq 2}} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$ is the *density* of G. One of the major open problems for multigraph edge colouring is the Goldberg-Seymour conjecture. It asks whether every multigraph satisfies $\chi(L(G)) \leq \max(\Delta(G)+1, D(G))$. Due to a classic theorem of Edmonds [36], this would imply that $\chi_f(L(G)) \leq \chi(L(G)) \leq \chi_f(L(G)) + 1$ for every multigraph (cf e.g. [34]).

There are two alternative characterizations of line graphs. First, a graph is the line graph of a (multi)graph if its edges can be partitioned into maximal cliques so that no vertex belongs to more than two such cliques. If moreover no two vertices are both in the same two cliques then it is the line graph of a simple graph.[83] We use this characterization in chapter 3. Second, line graphs of graphs are also characterized by nine forbidden induced subgraphs [8]. More precisely, a graph is the line graph of a simple graph iff it does not contain any of those nine graphs as an induced subgraph. The smallest and simplest of these nine forbidden graphs is the *claw*, which is the graph $K_{1,3}$, a vertex with three mutually nonadjacent neighbours. Thus the *claw-free* graphs form a well-studied generalization of line graphs. The chromatic number of a claw-free graph G is upper bounded by $\omega(G)^2$ and this cannot be improved to a linear function in $\omega(G)$. However, if additionally the graph contains at least one stable set of size 3 then the chromatic number is at most $2\omega(G)$ [28]. Recall furthermore that Reed's conjecture has been confirmed for claw-free graphs [72]. In chapter 3 we study a problem on claw-free graphs and deduce that it in fact reduces to the case of line graphs.

∞

From now on, we gradually shift focus from general background to the new contributions that are presented in this thesis. Please note the ∞−symbols, which mark the transition to a new subject. We first discuss the contents of chapter 5 on the chromatic number of the intersection graphs of several families of Jordan curves and Jordan regions, then chapters 3 and 4 on two approaches to the Erdős-Nešetřil conjecture on strong edge colouring, then chapter 2 on graph packing and the Bollobás-Eldridge-Catlin conjecture and finally chapter 6 on the dimension of the Incipient Infinite Cluster, a problem from percolation theory.

∞

Intersection graphs of Jordan regions and Jordan curves

Intersection graphs form another direction in which line graphs can be generalized. Given a family of subsets $\mathcal{F} := (F_i)_i$ of some base set, the intersection graph $G(\mathcal{F})$ of $\mathcal{F}$ is the graph with a vertex v_i for each F_i and an edge v_iv_j iff $F_i \cap F_j \neq \emptyset$. Well-studied examples include interval graphs (the intersection graph of a collection of intervals in $\mathbb{R}$) and intersection graphs of convex subsets of $\mathbb{R}^2$. Any finite graph G can be embedded in $\mathbb{R}^3$ such that the images of the edges are straight line segments that may only intersect in their endpoints. Now taking the embeddings of the edges as our family of subsets $\mathcal{F}$, we see that the line graph of G is precisely the intersection graph of $\mathcal{F}$. In chapter 5 we investigate the chromatic number of the intersection graph of certain families of Jordan regions and Jordan curves in the plane, and we obtain bounds in terms of the clique number. In particular, we investigate the intersection graph of a family of Jordan regions in the plane that pairwise intersect in at most one point. Using the discharging method, a degeneracy argument and list colouring, we show that the class of these graphs is $\chi-$bounded with binding function $f(x) = x + 327$. Furthermore, for this family we derive the stronger and sharp bound $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 1$ under the additional condition that $\omega(G(\mathcal{F})) \geq 490$. Since the binding function is of the form $x + C$ for some constant C, it follows from a recent theorem of Scott and Seymour (solving a conjecture of Gyárfás) that the class of complements of these intersection graphs is also χ-bounded [97]. Another result of ours (Corollary 5.1.11) implies that the intersection graph $G(\mathcal{F})$ of any family $\mathcal{F}$ of non-crossing Jordan curves in the plane is $(15.95 \cdot \omega(G(\mathcal{F}))-$colourable, and we provide ideas how the constant 15.95 may be improved. We do so in terms of a certain distance, where the distance between two Jordan curves roughly corresponds to the number of other Jordan curves separating them. The smaller the average distance between the Jordan curves in the family, the smaller the chromatic number of the intersection graph.

∞

Square of a graph

We will now discuss a type of colouring where equicoloured vertices need to be even further apart. The *distance* between two vertices $u, v \in V(G)$ is the minimum number t such that G contains a path between u and v on t edges[3]. The *square* G^2 of a graph G is the graph obtained from G by adding an edge between any two vertices that are at distance 2 in G. The chromatic number of G^2 reflects an efficient partition of the vertices in 'very stable' sets. The vertices of such a very stable set are mutually at distance at least three in G. As to the chromatic number, an easy upper bound is $\chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta^2 + 1$. It turns out that this is actually sharp up to a $(1 + o(1))$ factor (as $\Delta \to \infty$) and there even exist point-line incidence graphs of girth 6 (graphs for which the shortest cycle has length 6) that asymptotically attain this bound. Note

[3] If the vertices are in different components then the distance is ∞

that this contrasts with the ordinary chromatic numer, for which merely excluding triangles from G already reduces the best possible upper bound $\Delta(G)+1$ on $\chi(G)$ by a logarithmic factor $\ln(\Delta(G))$.

Square of a line graph

Having discussed line graphs, claw-free graphs and squares of graphs, we can now introduce one of the main topics of this thesis, namely the square of a line graph, $L(G)^2$. The *strong clique number* of a graph G is $\omega(L(G)^2)$. Similarly, the *strong chromatic index* of a graph G is defined as $\chi(L(G)^2)$. It is instructive to observe that each colour class in such a strong colouring must correspond to an *induced matching* of G, meaning that every two edges of the colour class are nonincident and have no edge between them. Despite elegant conjectures, analyzing optimal bounds for the strong chromatic index is deceivingly hard. This is well illustrated by a naive attempt to apply a degeneracy argument. Suppose we want to show that $\chi(L(G)^2) \le f(\Delta(G))$ for all graphs G and for some nondecreasing function f. One could try to derive the existence of an edge $e \in E(G)$ that has small degree $\le f(\Delta(G)) - 1$ in $L(G)^2$. Inductively we may then assume that $G-e$ has strong chromatic index $\le f(\Delta(G))$. There is at least one colour not appearing in the neighbourhood (with respect to $L(G)^2$) of e, 'thus' we can colour e and obtain the desired strong $f(\Delta(G))$-colouring. However, this inductive argument fails because $L(G-e)^2 \neq L(G)^2 - e$. In other words: it is not enough to complete the colouring on e, because after placing back e we also need to make sure that the edges incident to e mutually have different colours.

We are interested in how the strong chromatic index behaves in terms of the maximum degree of the graph. Let us first have a small excursion to the behaviour of a 'typical' random graph. In the Erdős-Renyi graph $G = G(n,p)$ on n vertices and with constant edge probability $0 < p < 1$ it holds with high probability that $\chi(L(G(n,p))^2) \le (1+o(1))\frac{3}{4}\frac{n^2 p}{\log_b n}$, where $b = 1/(1-p)$ [47]. This implies that with high probability, $\chi(L(G)^2) = O\left(\Delta(G)^2/\ln(\Delta(G))\right)$. In the sparse regime, where $np < \frac{1}{100}\sqrt{\log n/\log\log n}$, it holds with high probability that $\chi(L(G)^2) = \max\{d(u) + d(v) - 1 : (u,v) \in E(G)\} < 2\Delta(G)$.

Rather than further exploring the behaviour of a typical graph, we focus instead on the extremal question: what is the largest possible value among all graphs with fixed maximum degree Δ? According to a notorious conjecture of Erdős and Nešetřil (cf. [39]) it should hold for all graphs G with maximum degree Δ that

$$\chi(L(G)^2) \le \begin{cases} \frac{5\Delta^2}{4} & \text{if } \Delta \text{ is even;} \\ \frac{5\Delta^2 - 2\Delta + 1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

If true, this bound is sharp, as exemplified by so-called *blown-up* 5*-cycles*, which are graphs that can be obtained from a 5−cycle by 'blowing up' each vertex to a stable set of size $\lfloor \Delta/2 \rfloor$ and/or $\lceil \Delta/2 \rceil$ and replacing each edge of the 5−cycle with a complete bipartite graph between its corresponding stable sets.

Molloy and Reed [86] combined a structural estimate with a probabilistic colouring method to find a fixed but very small $\epsilon > 0$ such that $\chi(L(G)^2) \le (2-\epsilon)\Delta^2$ for all graphs. Bruhn and Joos [65] optimized this technique considerably and obtained $\chi(L(G)^2) \le 1.93 \cdot \Delta^2$

We approach the Erdős-Nešetřil conjecture from two directions. In chapter 3, we consider a strengthened form of the Erdős-Nešetřil conjecture. We ask whether for all claw-free graphs G it holds that $\chi(G^2) \le \frac{5}{4}\omega(G)^2$ if $\omega(G)$ is even, $\chi(G^2) \le \frac{1}{4}\left(5\omega(G)^2 - 2\omega(G) + 1\right)$ otherwise. Since line graphs are claw-free, this is a proper generalization of the Erdős-Nešetřil conjecture. In chapter 3 we investigate claw-free graphs with small clique number ($\omega(G) \in \{3, 4\}$) and show that this stronger conjecture holds iff the original Erdős-Nešetřil conjecture holds true, for (simple) graphs with maximum degree 3 respectively 4. Combining this with a result of Cranston [31], it follows that $\chi(G^2) \le 10$ for all claw-free graphs with clique number 3, which is sharp. For claw-free graphs with $\omega(G) \ge 6$, De Johannis de Verclos, Kang and Pastor (JKS [63]) have verified a similar reduction to line graphs of a multigraph. Thus, in terms of reducing to line graphs of multigraphs, only the case $\omega(G) = 5$ remains open. In their arguments, JKS first reduced the case of claw-free graphs to that of quasi-line graphs (which is a technical class of graphs between line graphs and claw-free graphs). Subsequently, they used a structure theorem of Chudnovksy and Seymour to reduce to the case of the line graph of a multigraph. In our arguments, we take a more direct approach, using a degeneracy argument. Such a degeneracy argument is however not as straighforward in G^2 as it is in G. Indeed, by adding a vertex v to G, two vertices that were not adjacent in G^2 can become adjacent in $(G + v)^2$. Thus a priori, for a degeneracy argument it does not suffice to find a vertex of small degree in G^2. We correct for this with a slightly altered greedy procedure, given by Lemma 3.3.1.

In chapter 4, we approach the Erdős-Nešetřil conjecture by studying a weakening rather than a strengthening. We consider the question of Faudree, Gýarfás, Schelp and Tuza [44], whether $\omega(L(G)^2) \le \frac{5}{4}\Delta^2$ holds for all graphs, and in Theorem 4.1.5 we prove it for all triangle-free graphs. Since the blown-up 5−cycles are triangle-free, this bound is sharp, and due to the fractional version of Reed's conjecture, we obtain as a corollary that $\chi_f(G) \le \frac{13}{8}\Delta(G)^2$ for all triangle-free G. For general graphs, the best known strong clique bound is $\omega(L(G)^2) \le 4/3 \cdot \Delta^2$, due to Faron and Postle [42].
In a classic paper, Faudree, Gýarfás, Schelp and Tuza [44] investigated how these parameters behave in bipartite graphs. For bipartite graphs G of maximum degree Δ, they proved that $\omega(L(G))^2 \le \Delta^2$ and conjectured that more generally $\chi(L(G)^2) \le \Delta^2$. These bounds are attained by the complete bipartite graph $K_{\Delta,\Delta}$. In chapter 4 we generalize this strong clique statement. In particular we show in Lemma 4.3.1 that all C_5−free multigraphs satisfy $\omega(L(G)^2) \le \sigma(G)^2/4 \le \Delta(G)^2$, where $\sigma(G) := \max_{xy \in E(G)} \deg(x) + \deg(y)$ denotes the *Ore-degree* of G. In conclusion we see that, with respect to the strong clique number, the (non)presence of C_5's distinguishes the class of bipartite graphs from the class of all graphs.

Observing that the (conjectured) extremal value of the strong clique number and the strong chromatic number are the same in the cases described up to now, one might wonder whether this phenomenon is dominant in the world of higher graph powers. We provide a negative answer. By a result of Mahdian [82], obtained with the probilistic method, there are graphs G of arbitrarily large girth such that $\chi(L(G)^2) \ge (\frac{1}{2} + o(1))\Delta(G)^2/\ln(\Delta(G))$. In contrast, we show that the strong clique number of high girth graphs is upper bounded by only a linear function of Δ. Indeed, for every integer $l \geq 3$, any graph G without cycles of order $\in \{l+1, l+2, l+3\}$ satisfies $\omega(L(G)^2) \leq l \cdot (\Delta - 1) + 2$. We conjecture that actually excluding only one even cycle C_{k+1} is sufficient for linear behaviour, yielding $\omega(L(G)^2) \leq k(\Delta - \frac{k-1}{2})$, and we prove it for C_4-free graphs. We identify *hairy cliques* (cliques with dangling edges attached to them) as extremal graphs for this conjectured bound.

Since we investigated the behaviour of the strong clique number under exclusion of all odd cycles (ie bipartite graphs) as well as under exlusion of one even cycle, it is natural to investigate what happens if we exclude both. It turns out that the extremal value of the strong chromatic index remains essentially unchanged, as there also exist *bipartite* graphs G of arbitrarily large girth with $\chi(L(G)^2) \geq (\frac{1}{2} + o(1))\Delta(G)^2/\ln(\Delta(G))$. On the other hand, we conjecture that in any bipartite graph of maximum degree $\Delta \geq 2$ that additionally contains no cycle of order $2k$, the strong clique number is at most $k\Delta + 1 - k$, which would be sharp if true. We provide evidence towards this conjecture in Theorem 4.1.13 and Corollary 4.1.15

Along the way we have also investigated how the strong clique number behaves in terms of the Hadwiger number and the maximum degree; see subsection 4.1.3. Roughly speaking, we prove that $\omega(L(G)^2)$ is at most the product of $h(G)$ and $\Delta(G)$, and we derive from Hadwiger's conjecture that this should also be the best possible bound for the strong chromatic index.

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Equitable colouring

An *equitable colouring* is a proper colouring of the vertices of a graph such that the sizes of the colour classes pairwise differ by at most one. Correspondingly, the *equitable chromatic number* $\chi_{eq}(G)$ of a graph G is the minimum integer k such that G can be equitably coloured with k colours. Unlike all types of colouring discussed up to now, the property of being equitably colourable with k colours is not monotone in k. That is, there are integers $k < l$ and a graph (on more than l vertices) that admits an equitable $k-$colouring but not an equitable $l-$colouring. For example: the complete bipartite graph $K_{2k+1,2k+1}$ has an equitable $2-$colouring but is not equitably $(2k+1)-$colourable.

In 1964, Erdős conjectured that every graph of maximum degree Δ has an equitable $(\Delta+1)-$colouring. In 1970 this was confirmed by Hajnal and Szemerédi [52]. In 2006, Kostochka and Kierstaed [77] found a much shorter and simpler proof, which moreover provided a polynomial time algorithm that constructs an equitable $(k+1)-$colouring of any graph G with maximum degree $\Delta(G) \leq k$. Several seemingly basic problems related to the equitable chromatic number remain open. For example, in spirit of Brooks' theorem, Chen, Lih and Wuh [76] conjectured that every connected graph with maximum degree $\Delta \geq 3$ distinct from $K_{\Delta+1}$ and $K_{\Delta,\Delta}$ is equitably $\Delta-$colourable. In chapter 2, we study a conjectured generalization of the Hajnal-Szemerédi Theorem, in terms of graph packing.

Packing of graphs

There exists a *packing* of two graphs G_1 and G_2 if one of them is a subgraph of the complement of the other graph. Many graph theoretical problems can be described in the language of packings. For three examples, let G be a graph on n vertices and note that (i): G contains a Hamiltonian cycle iff the cycle C_n packs with G, (ii): H is a subgraph of G iff H packs with the complement of G, (iii): if G has more than $ex(n, H)$ edges then it must pack with the complement of H.

Our favourite parameter, the chromatic number, can be described in the language of packing as well. The chromatic number of a graph G_1 equals the least integer k such that G_1 packs with some graph G_2 that consists of exactly k vertex-disjoint cliques and has the same number of vertices as G_1. Indeed, in such a packing, each clique of G_2 corresponds to a stable set -and thus to a colour class- of G_1.

In chapter 2, we study the following conjecture from the seventies. Bollobás, Eldridge and Catlin (BEC) conjectured that two graphs G_1 and G_2 on n vertices and with maximum degrees $\Delta(G_1)$ respectively $\Delta(G_2)$ pack if $(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1$. If true, it would constitute a significant generalization of the Hajnal-Szemerédi Theorem.

We consider the validity of the BEC-conjecture under the additional assumption that G_1 or G_2 has bounded codegree. More precisely, we prove for all $t \geq 2$ that, if G_1 does not have the complete bipartite graph $K_{2,t}$ as a subgraph and $\Delta(G_1) > 17t \cdot \Delta(G_2)$, then $(\Delta(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$ implies that G_1 and G_2 pack. As a corollary, we obtain that every $K_{2,t}-$free graph on n vertices and with maximum degree $\Delta \geq \sqrt{17t} \cdot \sqrt{n}$ has an equitable $\Delta-$colouring. As an application, we derive that the BEC conjecture also holds under the additional condition that both graphs do not contain a $4-, 6-$or $8-$cycle as a subgraph and one of the graphs has large enough maximum degree ($\geq 10^7$). The proofs are self-contained and of a combinatorial nature.

Recalling the Johansson-Molloy theorem that $\chi(G) = (1 + o(1))\Delta(G)/\ln(\Delta(G))$ for triangle-free graphs, it seems natural that this can be generalized to the setting of packing. In other words, is some condition of the form $\frac{\Delta(G_1)}{\ln(\Delta(G_1))}(\Delta(G_2) + 1) \leq cn$ for some constant $c > 0$ sufficient for G_1 and G_2 to pack if G_1 is triangle-free? Or more modestly, is it true that every triangle-free graph G has an equitable colouring with $O(\Delta(G)/\log(\Delta(G))$ colours?

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Percolation

In percolation theory, the main topic of study are the connected components of a random subgraph of a fixed graph. Given an (infinite) graph G and some $p \in [0, 1]$, a random subgraph is obtained as follows. Each edge is retained with probability p and discarded with probability $1 - p$, independently. The corresponding product probability measure is denoted $\mathbb{P}_p$. In this random graph, the *open cluster* $\mathcal{C}(v)$ of a vertex v is the connected component to which v belongs. By definition, the *critical probability* $p_c = p_c(G, v)$ of G and a vertex v is the supremum over all $p \in [0, 1]$ for which $\mathbb{P}_p$ (the cluster of v has finite size) = 1. If the graph is vertex-transitive, p_c only depends on G. The value of p_c is far from known, except in special cases where certain symmetries of G and/or its dual can be exploited.

A major open question is: for which graphs is p_c strictly smaller than 1? An easy example of a graph with $p_c = 1$ is the graph with vertices $\mathbb{Z}$ and edges between subsequent integers. Benjamini and Schramm conjectured [9] that $p_c < 1$ for all graphs with isoperimetric dimension $d_i > 1$, where $d_i := \sup \left\{ d > 0 \mid \inf_{\emptyset \neq S \subseteq V(G)} \frac{|\partial S|}{|S|^{(d-1)/d}} > 0 \right\}$. Here ∂S denotes the boundary of S, the set of vertices in $V(G) \backslash S$ that have a neighbour in S. This conjecture has been confirmed in several special cases, among which are (roughly) transitive graphs of polynomial growth[23].

From now on, we consider percolation on the graph with vertices $\mathbb{Z}^d$ and edges only between nearest neighbours. This is a well-studied classic model. For dimension $d = 2$, it is known that $p_c = 1/2$ but p_c is unknown and hard to approximate for larger d.

The behaviour of the random graph drastically changes around p_c, since $\mathbb{P}_p$ (there exists *some* cluster of infinite size) transitions from 0 if $p < p_c$, to 1 if $p > p_c$. Moreover, it can be shown that (for $p > p_c$) the infinite cluster is *unique* with probability 1. What happens at p_c itself remains an important open question, in general. For nearest-neighbour edge percolation on $\mathbb{Z}^d$, it is conjectured that

$$\theta(p) := \mathbb{P}_p \text{ (the cluster of the origin has infinite size)}$$

equals 0 for $p = p_c$, which would imply the continuity of $\theta(p)$ as a function of p. This has been confirmed for $d = 2$ and in *high dimensions* ($d \geq 11$), but remains in particular elusive for $d = 3$.

In chapter 6, we focus on what happens in high dimensions at p_c. Even though $\theta(p_c) = 0$ there, so that there should be no cluster of infinite size, there exists a natural probability measure that in some sense conditions on the event that the cluster of the origin *is* infinitely large. Under this alternative 'Incipient Infinite Cluster' measure, we show that the cluster of the origin is almost surely a 4−dimensional object, in the sense that in a box of radius r around the origin, the number of vertices belonging to the cluster is of order r^4. To put this in perspective, for $p < p_c$ the cluster of the origin is $\mathbb{P}_p$−almost surely 0−dimensional, while for $p > p_c$ it is $\mathbb{P}_p$-almost surely d-dimensional.

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Notation

Basic knowledge of graph theory and probability theory is assumed. Notation can differ slightly from chapter to chapter and therefore will be explained in each chapter separately. In particular, in chapter 3, $N_G(S)$ denotes the open neighbourhood $\bigcup_{s \in S} N_G(s) \backslash S$ of a set of vertices S in a graph G, while in the other chapters $N_G(S)$ refers to the closed neighbourhood $\bigcup_{s \in S} N_G(s)$.

Source material

Chapter 2 is based on the papers [20] and [22], as well as some small new results. Chapters 3, 5 and 6 are almost literally the papers [21],[19],[18] respectively. Finally, chapter 4 is part of a paper in preparation (joint with Ross J. Kang and François Pirot). The result in the appendix has not been submitted for publication.

Chapter 2

On the Bollobás–Eldridge–Catlin conjecture

Two graphs G_1 and G_2 on n vertices are said to *pack* if there exist injective mappings of their vertex sets into $[n]$ such that the images of their edge sets are disjoint. A long-standing conjecture due to Bollobás and Eldridge and, independently, Catlin, asserts that, if $(\Delta(G_1)+1)(\Delta(G_2)+1) \le n+1$, then G_1 and G_2 pack. We consider the validity of this assertion under the additional assumption that G_1 or G_2 has bounded codegree. In particular, we prove for all $t \ge 2$ that, if G_1 contains no copy of the complete bipartite graph $K_{2,t}$ and $\Delta(G_1) > 17t \cdot \Delta(G_2)$, then $(\Delta(G_1)+1)(\Delta(G_2)+1) \le n+1$ implies that G_1 and G_2 pack. We also provide a mild improvement if moreover G_2 contains no copy of the complete tripartite graph $K_{1,1,s}$, $s \ge 1$.

As an application, we derive that the BEC conjecture also holds under the additional condition that both graphs don't contain a $4-, 6-$or $8-$cycle as a subgraph and one of the graphs has large enough maximum degree ($\ge 10^7$).

Finally, we derive a bound that interpolates between a result of Eaton and a result of Sauer and Spencer. For $0 \le q \le \lfloor n/2 + 1 - \Delta(G_1) - \Delta(G_2) \rfloor$, we prove that if $\Delta(G_1)\Delta(G_2) < n/2 + q$ then G_1 and G_2 admit a degree ≤ 1 near-packing with at most q common edges.

2.1 Packing graphs of bounded codegree

Let G_1 and G_2 be graphs on n vertices. (All graphs are assumed to have neither loops nor multiple edges.) We say that G_1 and G_2 *pack* if there exist injective mappings of their vertex sets into $[n] = \{1, \dots, n\}$ so that their edge sets have disjoint images. Equivalently, G_1 and G_2 pack if G_1 is a subgraph of the complement of G_2. The *maximum codegree* $\Delta^\wedge(G)$ of a graph G is the maximum over all vertex pairs of their common degree, i.e. $\Delta^\wedge(G) < t$ if and only if G contains no copy of the complete bipartite graph $K_{2,t}$. The *maximum adjacent codegree* $\Delta^\triangle(G)$ of G is the maximum over all pairs of *adjacent* vertices of their common degree, i.e. $\Delta^{\triangle}(G) < s$ if and only if G contains no copy of the complete tripartite graph $K_{1,1,s}$. Clearly, $\Delta^{\triangle}(G) \le \Delta^{\wedge}(G)$ always. We let Δ_1 and Δ_2 denote the maximum degrees of G_1 and G_2, respectively, and $\Delta_1^{\wedge}$ and $\Delta_2^{\triangle}$ the corresponding maximum (adjacent) codegrees. We provide sufficient conditions for G_1 and G_2 to pack in terms of Δ_1, Δ_2, $\Delta_1^{\wedge}$, $\Delta_2^{\triangle}$.

For integers $t \ge 2$ and $\Delta_2 \ge 1$, we define

$$\alpha^*(t, \Delta_2) := \frac{1}{2}(2 + \gamma + \sqrt{4\gamma + \gamma^2}), \quad \text{where} \quad \gamma = \frac{\Delta_2}{\Delta_2 + 1} \cdot \frac{t-1}{t}.$$

Note $\alpha^* = \alpha^*(t, \Delta_2)$ is the larger solution to the equation $(\alpha - 1)^2 - \gamma\alpha = 0$ and $\frac{1}{8}(9 + \sqrt{17}) \le \alpha \le \frac{1}{2}(3 + \sqrt{5})$.

Theorem 2.1.1. *Let G_1 and G_2 be graphs on n vertices with respective maximum degrees Δ_1 and Δ_2. Let $\Delta_1^{\wedge}$ be the maximum codegree of G_1. Let $t \ge 2$ be an integer and let $\alpha > \alpha^* = \alpha^*(t, \Delta_2)$ and $0 < \epsilon < 1/2$ be reals. Then G_1 and G_2 pack if $\Delta_1^{\wedge} < t$ and n is larger than each of the following quantities:*

$$\left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2, \tag{2.1}$$

$$(2\alpha t + 2) \cdot \Delta_2 + ((2\alpha + 1)t - 1) \cdot \Delta_2^2 + (1 - \epsilon) \cdot \Delta_1\Delta_2, \tag{2.2}$$

$$1 + \left(2 + \frac{\epsilon}{1 - 2\epsilon}\right) \cdot \Delta_2 + \Delta_1\Delta_2, \quad \textit{and} \tag{2.3}$$

$$\left(t + \frac{3-\epsilon}{2}\right) \cdot \Delta_2 + \frac{3-\epsilon}{2}(t-1) \cdot \Delta_2^2 + \frac{1+\epsilon}{2} \cdot \Delta_1\Delta_2. \tag{2.4}$$

Theorem 2.1.2. *Let G_1 and G_2 be graphs on n vertices with respective maximum degrees Δ_1 and Δ_2. Let $\Delta_1^{\wedge}$ be the maximum codegree of G_1 and $\Delta_2^{\triangle}$ the maximum adjacent codegree of G_2. Let $s \ge 1$ and $t \ge 2$ be integers and let $\alpha > \alpha^* = \alpha^*(t, \Delta_2)$ be real. Then G_1 and G_2 pack if $\Delta_1^{\wedge} < t$, $\Delta_2^{\triangle} < s$, and n is larger than both of the following quantities:*

$$\left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2 \quad \textit{and} \tag{2.5}$$

$$(2 + 2\alpha t) \cdot \Delta_2 + (s - 1) \cdot \Delta_1 + ((2\alpha + 1)t - 1) \cdot \Delta_2^2. \tag{2.6}$$

For better context, we compare Theorems 2.1.1 and 2.1.2 to a line of work on graph packing that was initiated in the 1970s [12, 24, 25, 94]. The following is a central problem in the area.

Conjecture 2.1.3 (Bollobás and Eldridge [12] and Catlin [25])**.** *Let G_1 and G_2 be graphs on n vertices with respective maximum degrees Δ_1 and Δ_2. Then G_1 and G_2 pack if $(\Delta_1 + 1)(\Delta_2 + 1) \le n + 1$.*

If true, the statement would be sharp and would significantly generalise a celebrated result of Hajnal and Szemerédi [52] on equitable colourings. Sauer and Spencer [94] showed that $2\Delta_1\Delta_2 < n$ is a sufficient condition for G_1 and G_2 to pack, which is seen to be sharp when one of the graphs is a perfect matching. Thus far the Bollobás–Eldridge–Catlin (BEC) conjecture has been confirmed in the following special cases: $\Delta_1 = 2$ [3]; $\Delta_1 = 3$ and n sufficiently large [33]; G_1 bipartite and n sufficiently large [32]; and G_1 d-degenerate, $\Delta_1 \geq 40d$ and $\Delta_2 \geq 215$ [15]. We would also like to highlight the following three results that can be considered approximate forms of the BEC conjecture. (a) The condition $(\Delta_1 + 1)(\Delta_2 + 1) \leq 3n/5 + 1$ is sufficient for G_1 and G_2 to pack, provided that $\Delta_1, \Delta_2 \geq 300$ [68]. (b) The BEC condition is sufficient for G_1 and G_2 to admit a 'near packing' in that the subgraph induced by the intersection of their images has maximum degree at most 1 [35]. (c) If G_2 is chosen as a binomial random graph of parameters n and p such that np in place of Δ_2 satisfies the BEC condition, then G_1 and G_2 pack with probability tending to 1 as $n \to \infty$ [13].

Corollary 2.1.4. *Let $G_1, G_2, \Delta_1, \Delta_2$ and $\Delta_1^\wedge$ be as before. Let $t \geq 2$ be an integer. Then G_1 and G_2 pack if $\Delta_1\Delta_2 + \Delta_1 \leq n + 1$ and $\Delta_1^\wedge < t$ and $\Delta_1 > 17t \cdot \Delta_2$.*

Proof. Choose $\epsilon = (2t - 2)/(4t - 3)$ and $\alpha = 3$ in Theorem 2.1.1. Using that $\Delta_1 > 17t\Delta_2 > \frac{(4t-3)(7t-1)}{2t-2} \cdot \Delta_2$, it follows that $\max((2.1), (2.2), (2.3), (2.4)) \leq (\Delta_1 + 1)(\Delta_2 + 1) - 1 \leq n$. So G_1 and G_2 pack. □

We have the following results concerning the BEC conjecture.

Corollary 2.1.5. *Given an integer $t \geq 2$, the BEC conjecture holds under the additional condition that the maximum codegree $\Delta_1^\wedge$ of G_1 is less than t and $\Delta_1 > 17t \cdot \Delta_2$.*

We were unable to avoid the linear dependence on Δ_2 in the lower bound condition on Δ_1. Although we have not seriously attempted to optimise the factor $17t$ above, Theorem 2.1.2 improves on this factor under the additional assumption that $\Delta_2^\triangle$ is bounded, as exemplified by the following corollary.

Corollary 2.1.6. *Given an integer $t \geq 2$, the BEC conjecture holds under the additional condition that the maximum codegree $\Delta_1^\wedge$ of G_1 is less than t, G_2 is triangle-free, and $\Delta_1 > (4 + \sqrt{5})t \cdot \Delta_2$.*

Proof. Choose $\alpha = \frac{1}{4t}(6t + 1 + \sqrt{20t^2 + 4t + 1})$ and $s = 1$ in Theorem 2.1.2. Using that $t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2-\alpha} - 1 = (2\alpha + 1)t - 1$ and that $\Delta_1 > (4 + \sqrt{5})t \cdot \Delta_2 > ((2\alpha + 1)t - 1) \cdot \Delta_2$, it follows that $\max((2.5), (2.6)) \leq (\Delta_1 + 1)(\Delta_2 + 1) - 1 \leq n$. So G_1 and G_2 pack. □

Application to equitable colourings

An *equitable colouring* is a proper vertex-colouring for which the sizes of the colour classes pairwise differ by at most 1. By taking G_2 to be a collection of (nearly) equal-sized cliques, Corollary 2.1.4 implies that, if G is a $K_{2,t}$-free graph of maximum degree Δ with $\Delta \geq \sqrt{17t} \cdot \sqrt{n}$, then the equitable chromatic number of G is at most Δ. Note that this result cannot be obtained by the result of Hajnal and Szemerédi [52], which says that the equitable chromatic number of every graph is at most $\Delta + 1$.

Corollary 2.1.7. *Let $t \geq 2$ be an integer. Let G_1 be a $K_{2,t}-$free graph on n vertices with maximum degree Δ_1, such that $\Delta_1 \geq \sqrt{17t} \cdot \sqrt{n}$. Then there is a Δ_1-equitable colouring of G_1.*

Proof. In general, n is not divisible by Δ_1, so define the remainder $r = \Delta_1 \lceil n/\Delta_1 \rceil - n$ and the augmented graph $G_1^+ = G_1 + K_r$, where K_r is a complete graph. Write $n^+ = n + r$ for the order of G_1^+. Note that $0 \le r \le \Delta_1 - 1$, that $\Delta(G_1^+) = \Delta_1$ and that n^+ is divisile by Δ_1.

Now choose G_2 to be a collection of Δ_1 disjoint cliques with degree Δ_2, such that $\Delta_1\Delta_2 + \Delta_1 = n^+$. Note that $\Delta_1^\wedge < t$, because G_1 is $K_{2,t}$-free. Furthermore,

$$\begin{aligned}
\Delta_1 \ge \sqrt{17t} \cdot \sqrt{n} \quad &\Rightarrow \quad \Delta_1^2 + 17t > 17tn \\
&\Leftrightarrow \quad \Delta_1^2 > 17t \cdot (n + \Delta_1 - 1 - \Delta_1) \\
&\Rightarrow \quad \Delta_1^2 > 17t \cdot (n + r - \Delta_1) \\
&\Leftrightarrow \quad \Delta_1 > 17t \cdot (n^+/\Delta_1 - 1) \\
&\Leftrightarrow \quad \Delta_1 > 17t \cdot \Delta_2.
\end{aligned}$$

So we can apply Corollary 2.1.4 to conclude that G_1^+ and G_2 pack. In such a packing, each (maximal) clique in G_2 corresponds to an independent set in G_1. We assign a colour to each of these Δ_1 disjoint and equal-sized independent sets. Thus we have derived the existence of a colouring of G_1^+ with Δ_1 equal-sized colour classes. Since each vertex in K_r must have a different colour, this induces an equitable colouring of G_1 with Δ_1 colours. □

Possible generalizations

The BEC conjecture notwithstanding, naturally one might wonder whether Theorem 2.1.1, or rather Corollary 2.1.5, could be improved according to a weaker form of the BEC condition, as was the case for d-degenerate G_1 [15]. In other words, it would be interesting to improve upon the $\Omega(\Delta_1\Delta_2)$ terms appearing in each of (2.1)–(2.4). We leave this to further study, but point out the following constructions where G_1 has low maximum codegree, which mark boundaries for this problem.

- When n is even, there are non-packable pairs (G_1, G_2) of graphs where G_1 is a perfect matching (so $\Delta_1^\wedge = 0$) and $2\Delta_1\Delta_2 = n$, cf. [67].
- Bollobás, Kostochka and Nakprasit [14] exhibited a family of non-packable pairs (G_1, G_2) of graphs where G_1 is a forest (so $\Delta_1^\wedge = 1$) and $\Delta_1 \ln \Delta_2 \ge cn$ for some $c > 0$.
- If $\Delta^\wedge(G) = 1$, then the chromatic number of G satisfies $\chi(G) = O(\Delta(G)/\ln \Delta(G))$ as $\Delta(G) \to \infty$, and there are standard examples having arbitrarily large girth that show this bound to be sharp up to a constant factor, cf. [87, Ex. 12.7]. Since the equitable chromatic number is at least the chromatic number, these examples moreover yield non-packable pairs (G_1, G_2) of graphs having $\frac{\Delta_1}{\ln \Delta_1}(\Delta_2 + 1) \ge cn$ for some $c > 0$ and $\Delta_1^\wedge = 1$.

Since the examples can also have the maximum adjacent codegree $\Delta_1^\vartriangle$ being zero, this last remark hints at another natural line to pursue, which could significantly extend both the result of Csaba [32] and a result of Johansson [64]. If Δ_1 is large enough and G_1 is triangle-free, is some condition of the form $\frac{\Delta_1}{\ln \Delta_1}(\Delta_2 + 1) \le cn$ for some constant $c > 0$ sufficient for G_1 and G_2 to pack?

Structure

In the next section, we provide some notation and preliminary observations. In Section 2.3, we discuss the common features of a hypothetical critical counterexample to one of our theorems. In Section 2.4, we prove Theorems 2.1.1 and 2.1.2. We conclude with some remarks about the results, proofs and further possibilities.

2.2 Notation and preliminaries

Here we introduce some terminology which we use throughout. We often call G_1 the *blue* graph and G_2 the *red* graph. We treat the injective vertex mappings as labellings of the vertices from 1 to n. However, rather than saying, "the vertex in G_1 (or G_2) corresponding to the label i", we often only say, "vertex i", since this should never cause any confusion. Our proofs rely on accurately specifying the neighbourhood structure as viewed from a particular vertex. Let $i \in [n]$. The *blue neighbourhood* $N_1(i)$ *of* i is the set $\{j \mid ij \in E(G_1)\}$ and the *blue degree* $\deg_1(i)$ *of* i is $|N_1(i)|$. The *red neighbourhood* $N_2(i)$ and *red degree* $N_2(i)$ are defined analogously. For $j \in [n]$, a *red–blue-link (or* 2–1*-link) from* i *to* j is a vertex i' such that $ii' \in E(G_2)$ and $i'j \in E(G_1)$. The *red–blue-neighbourhood* $N_1(N_2(i))$ *of* i is the set $\{j \mid \exists$ red–blue-link from i to $j\}$. A *blue–red-link (or* 1–2*-link)* and the *blue–red-neighbourhood* $N_2(N_1(i))$ are defined analogously.

In search of a certificate that G_1 and G_2 pack, without loss of generality, we keep the vertex labelling of the blue graph G_1 fixed, and permute only the labels in the red graph G_2. This can be thought of as "moving" the red graph above a fixed ground set $[n]$. In particular, we seek to avoid the situation that there are $i, j \in [n]$ for which ij is an edge in both G_1 and G_2 — in this situation, we call ij a *purple* edge induced by the labellings of G_1 and G_2. So G_1 and G_2 pack if and only if they admit a pair of vertex labellings that induces no purple edge. In our search, we make small cyclic sub-permutations of the labels (of G_2), which are referred to as follows. For $i_0, \dots, i_{\ell-1} \in [n]$, a $(i_0, \dots, i_{\ell-1})$*-swap* is a relabelling of G_2 so that for each $k \in \{0, \dots, \ell-1\}$ the vertex labelled i_k is re-assigned the label $i_{k+1 \bmod \ell}$. In fact, we shall only require swaps having $\ell \in \{1, 2\}$. The following observation describes when a swap could be helpful in the search for a packing certificate. This is identical to Lemma 1 in [68].

Lemma 2.2.1. *Let $u_0, \dots, u_{\ell-1} \in [n]$. For every $k, k' \in \{0, \dots, \ell-1\}$, suppose that there is no red–blue-link from u_k to $u_{k+1 \bmod \ell}$ and that, if $u_k u_{k'} \in E(G_2)$, then $u_{(k+1 \bmod \ell)} u_{(k'+1 \bmod \ell)} \notin E(G_1)$. Then there is no purple edge incident to any of $u_0, \dots, u_{\ell-1}$ after a $(u_0, \dots, u_{\ell-1})$-swap.* □

We will use a classic extremal set theoretic result to upper bound the size of certain vertex subsets.

Lemma 2.2.2 (Corrádi [30])**.** *Let $A_1, \dots, A_N$ be k-element sets and X be their union. If $|A_i \cap A_j| \le t-1$ for all $i \neq j$, then $|X| \ge k^2 N/(k + (N-1)(t-1))$.*

Proof. Given $x \in X$, let $d(x) := \{i \in \{1, \dots, N\} \mid x \in A_i\}$ denote the number of subsets it belongs to. Then for each $i \in \{1, \ldots, N\}$,

$$\sum_{x \in A_i} d(x) = \sum_{j=1}^{N} |A_i \cap A_j| = |A_i| + \sum_{j \neq i} |A_i \cap A_j| \leq k + (N-1)(t-1).$$

Summing over all sets A_i and using Jensen's inequality yields

$$\sum_{i=1}^{N} \sum_{x \in A_i} d(x) = \sum_{x \in X} d(x)^2 \geq \frac{1}{|X|} \left(\sum_{x \in X} d(x) \right)^2 = \frac{1}{|X|} \left(\sum_{i=1}^{N} |A_i| \right)^2 = \frac{(Nk)^2}{|X|}.$$

Combining these two inequalities, we obtain $(Nk)^2 \leq N \cdot |X| \cdot (k + (N-1)(t-1))$. □

In particular, this implies the following.

Corollary 2.2.3. *Let $A_1, \ldots, A_N$ be size $\geq k$ subsets of a set X. If $k^2 > (t-1) \cdot |X|$ and $|A_i \cap A_j| \leq t-1$ for all $i \neq j$, then*

$$N \leq |X| \cdot \frac{k - (t-1)}{k^2 - (t-1) \cdot |X|}.$$

Proof. Consider arbitrary subsets $A_1^* \subset A_1, \ldots, A_N^* \subset A_N$ of size k. An application of Corrádi's lemma to $A_1^*, \ldots, A_N^*$ yields that $|X| \geq k^2 \cdot N/(k + (N-1)(t-1))$, which is easily seen to be equivalent to $(k^2 - (t-1) \cdot |X|) \cdot N \leq (k-t+1) \cdot |X|$. The corollary follows after dividing both sides of the inequality by $k^2 - (t-1) \cdot |X|$. Note that this division does not cause a sign change because of the assumption that $k^2 > (t-1) \cdot |X|$. □

In particular (roughly speaking): if we have subsets of a set X that have pairwise uniformly bounded overlap and are of size $\Omega(\sqrt{|X|})$, then there are at most $O(\sqrt{|X|})$ of them. This is the property that we will use.

2.3 Hypothetical critical counterexamples

The overall proof structure we use for both theorems is the same, and in this section we describe common features and some further notation. Suppose the theorem (one of Theorem 2.1.1 or 2.1.2) is false. Then there must exist a counterexample, that is, a pair (G_1, G_2) of non-packable graphs on n vertices that satisfy the conditions of the theorem.

Moreover, we may assume that (G_1, G_2) is a *critical* pair in the sense that G_2 is edge-minimal among all counterexamples. In other words, G_1 and $G_2 - e$ pack for *any* $e \in E(G_2)$. There is no loss of generality, since the removal of an edge from G_2 increases neither Δ_2 nor $\Delta_2^{\triangle}$ and obviously affects none of Δ_1, $\Delta_1^{\wedge}$ and n, thus maintaining the required conditions.

Now choose *any* edge $e = uv \in E(G_2)$. Criticality implies that there is a pair of labellings of G_1 and G_2 such that e is the *unique* purple edge, for otherwise G_1 and $G_2 - e$ do not pack. Let us fix such a pair of labellings so that we can further describe the neighbourhood structure as viewed from u (or v). Estimation of the sizes of subsets

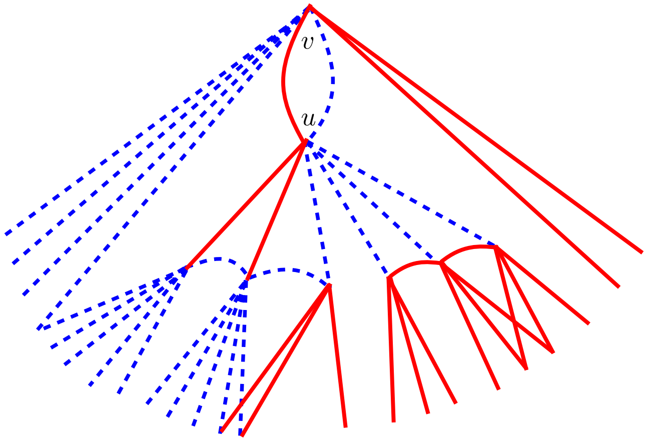

Figure 2.1: All vertices (except possibly v) are reachable by a link from u (Claim 2.3.1).

in this neighbourhood structure is our main method for deriving upper bounds on n that in turn yield the desired contradiction from which the theorem follows.

We need the definition of the following vertex subsets (which are analogously defined for v also):

$$\begin{aligned} A(u) &:= N_2(N_1(u)) \setminus (N_1(u) \cup N_2(u) \cup N_1(N_2(u))), \\ B(u) &:= N_1(N_2(u)) \setminus (N_1(u) \cup N_2(u) \cup N_2(N_1(u))), \\ A^*(u) &:= N_2(N_1(u)) \setminus (N_2(u) \cup N_1(N_2(u))), \text{ and} \\ N_1^*(u) &:= N_1(u) \cap (N_1(N_2(u)) \setminus (N_2(u) \cup N_2(N_1(u)))). \end{aligned}$$

One justification for specifying the above subsets is that the following two claims (which are essentially Claims 1 and 2 in [68]) hold.

Claim 2.3.1. *For all $w \in [n] \setminus \{v\}$, there is a red–blue-link or a blue–red-link from u to w.*

Proof. If not, then by Lemma 2.2.1, a (u, w)-swap yields a new labelling such that uv is not purple anymore and no new purple edges are created. Thus G_1 and G_2 pack, a contradiction. See Figure 2.1. □

Claim 2.3.2. *For all $a \in A^*(u)$ and $b \in B(u)$, there is a red–blue-link from a to b.*

Proof. Since $B(u) \cap N_1(u) = B(u) \cap N_2(u) = \emptyset$ and $A^*(u) \cap N_2(u) = \emptyset$, we have that $bu \notin E(G_1) \cup E(G_2)$ and $ua \notin E(G_2)$. Furthermore, since $A^*(u) \cap N_1(N_2(u)) = B(u) \cap N_2(N_1(u)) = \emptyset$, there is no red–blue-link from u to a or from b to u. Now suppose that there is also no red–blue-link from a to b. Then it follows from Lemma 2.2.1 that after a (u, a, b)-swap there is no purple edge incident to any of u, a, b, which implies that there is no purple edge at all. So we have obtained a packing of G_1 and G_2, a contradiction. □

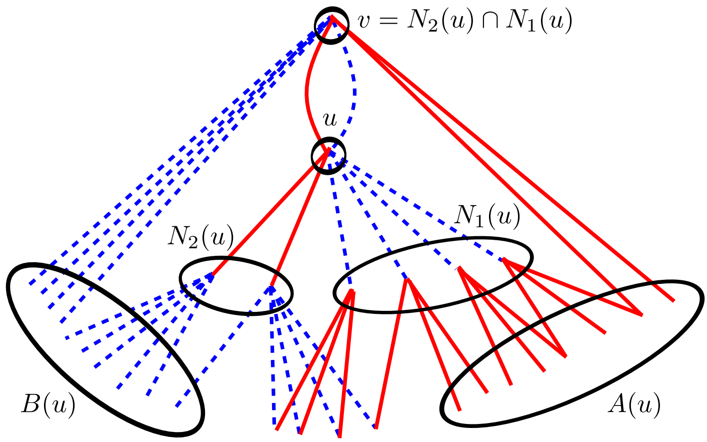

Figure 2.2: The neighbourhood structure of a hypothetical critical counterexample, as seen from u.

In the next claim, we list three upper bounds on the total number n of vertices in terms of the sizes of the vertex subsets defined above. In the proofs of Theorems 2.1.1 and 2.1.2, we consider several cases for which we prove at least one of these upper bounds to be small enough for a contradiction with the assumed lower bounds on n.

Claim 2.3.3. *The total number n of vertices is at most each of the following quantities:*

1. $|N_2(u)| + |A^*(u)| + |N_1(N_2(u))|$,
2. $|N_1^*(u)| + |N_2(u)| + |B(u)| + |N_2(N_1(u))|$,
3. $|A^*(v)| + |A^*(u)| + |\,(N_2(u) \cup N_1(N_2(u))) \cap (N_2(v) \cup N_1(N_2(v)))\,|$.

Proof. In all cases, $[n]$ equals the union of the neighbourhood sets that occur in the upper bound.

1 The union of $N_2(u)$, $A^*(u)$ and $N_1(N_2(u))$ covers $\{v\} \cup N_2(N_1(u)) \cup N_1(N_2(u))$, which by Claim 2.3.1 equals $[n]$.

2 The union of $N_1^*(u)$, $N_2(u)$, $B(u)$ and $N_2(N_1(u))$ covers $\{v\} \cup N_2(N_1(u)) \cup N_1(N_2(u))$, which equals $[n]$.

3 By the proof of (i), $[n]$ is the union of $A^*(u)$ and $N_2(u) \cup N_1(N_2(u))$ as well as the union of $A^*(v)$ and $N_2(v) \cup N_1(N_2(v))$. It follows that $[n]$ also is the union of $A^*(u)$, $A^*(v)$ and $(N_2(u) \cup N_1(N_2(u))) \cap (N_2(v) \cup N_1(N_2(v)))$. □

The reason for working with $N_1^*(u)$ and $A^*(u)$ rather than the simpler sets $N_1(u)$ and $A(u)$ is the following. Under the requirement that the codegree $\Delta_1^\wedge$ of G_1 is less than t, we can upper bound $|N_1^*(u)|$ entirely in terms of Δ_2. This is sharper than the trivial bound $|N_1(u)| \leq \Delta_1$ because we work under conditions with Δ_1 rather larger than Δ_2. Similarly, since $N_1^*(u) \subset N_1(u)$, we need to compensate for the loss of covered vertices by working with the slightly enlarged set $A^*(u)$, rather than $A(u)$. The following claims use the condition $\Delta_1^\wedge < t$ (which is assumed by both theorems).

Claim 2.3.4. $|N_1^*(u)| \leq (t-1) \cdot \Delta_2$.

Proof. Suppose $|N_1(u) \cap N_1(N_2(u))| \geq (t-1) \cdot \Delta_2 + 1$, then there is at least one $x \in N_2(u)$ such that $|N_1(u) \cap N_1(x)| \geq \frac{1}{|N_2(u)|} \cdot ((t-1) \cdot \Delta_2 + 1) > t-1$, which contradicts $\Delta_1^\wedge < t$. □

The following claim (in combination with Corrádi's lemma) is useful for an upper bound on $|B(u)|$ that is only linear in Δ_2, provided that $|A^*(u)|$ is at least quadratic in Δ_2. See Case 1 in the proof of Theorem 2.1.1.

Claim 2.3.5. *For any $b \in B(u)$, $|N_1(b) \cap A^*(u)| \geq |A^*(u)|/\Delta_2 - t(\Delta_2 + 1)$.*

Proof. For all $b \in N_1(N_2(u))$ it holds that $|N_1(b) \cap N_1(N_2(u))| \leq (t-1) \cdot |N_2(u)| \leq (t-1) \cdot \Delta_2$. Indeed, otherwise there would exist a blue copy of $K_{2,t}$ in the graph induced by $N_1(N_2(u)) \cup N_2(u)$. Similarly, $|N_1(b) \cap N_1(u)| \leq t$ and $|N_1(b) \cap N_2(u)| \leq \Delta_2$. So for every $b \in N_1(N_2(u))$, at most $t \cdot (\Delta_2 + 1)$ blue neighbours of b are in $[n] \setminus A(u)$. So in particular, for every $b \in B(u)$, at most $t \cdot (\Delta_2 + 1)$ blue neighbours of b are in $[n] \setminus A^*(u)$.

Using Claim 2.3.2 and the fact that each blue neighbour of a fixed $b \in B(u)$ has at most Δ_2 red neighbours in $A^*(u)$, we see that every $b \in B(u)$ has at least $\lceil |A^*(u)|/\Delta_2 \rceil$ blue neighbours, and thus at least $|A^*(u)|/\Delta_2 - t(\Delta_2 + 1)$ blue neighbours in $A^*(u)$. □

2.4 Proofs

2.4.1 Proof of Theorem 2.1.1

Suppose the theorem is false. Consider a critical counterexample, a pair of non-packable graphs (G_1, G_2), with G_2 edge-minimal, satisfying the constraints of the theorem. We distinguish three cases, for each of which we derive an upper bound on n, given by one of the inequalities (2.8), (2.10) and (2.16). At least one of these three inequalities should hold, so together they contradict the condition that max ((2.8), (2.10), (2.16)) = max((2.1), (2.2), (2.3), (2.4)) $< n$, thus proving the theorem.

1. There exists a vertex $u \in [n]$ and there are labellings of G_1 and G_2 such that u is incident to the unique purple edge and $|A^*(u)| \geq \alpha t \cdot \Delta_2(\Delta_2 + 1)$.

2. Case 1 does not hold and furthermore $|N_2(u) \cap N_2(v)| < (1-\epsilon) \cdot \Delta_2$ for some edge $uv \in E(G_2)$.

3. Case 1 does not hold and $|N_2(u) \cap N_2(v)| \geq (1-\epsilon) \cdot \Delta_2$ for every $uv \in E(G_2)$.

We now proceed with deriving upper bounds on n for each of these three cases.

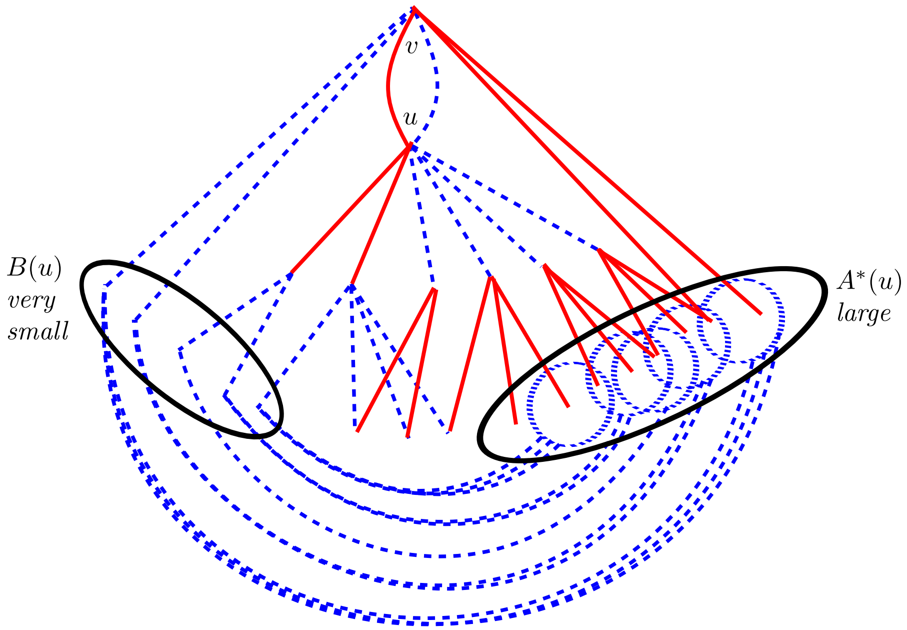

Figure 2.3: A depiction of Case 1 of Theorem 2.1.1, that $|A^*(u)| = \Omega(\Delta_2^2)$ implies $|B(u)| = O(\Delta_2)$.

Bound for Case 1. Choose a vertex $u \in [n]$ and labellings of G_1 and G_2 such that u is incident to the unique purple edge and $|A^*(u)| \geq \alpha t \cdot \Delta_2(\Delta_2 + 1)$. See Figure 2.3 for a depiction of the argumentation in this case. From now on, we write $k := |A^*(u)|/\Delta_2 - t(\Delta_2 + 1)$. Our first tool is Claim 2.3.5, which yields that all $b \in B(u)$ satisfy $|N_1(b) \cap A^*(u)| \geq k$. Note that $k \geq 1$, since $\alpha > 1$. Our second tool is Corrádi's lemma, or rather Corollary 2.2.3, which we apply with $X = A^*(u)$ and $N = |B(u)|$ and with size $\geq k$ subsets $A_1, \ldots, A_N \subset X$ given by $N_1(b) \cap A^*(u)$, for all $b \in B(u)$. Note that $|A_i \cap A_j| \leq t - 1$ for all $i \neq j$, or else there would be a blue copy of $K_{2,t}$.

In order to apply Corollary 2.2.3, we need to check that its condition $k^2 > (t - 1) \cdot |A^*(u)|$ holds. For that, we write $\beta := |A^*(u)|/(t\Delta_2(\Delta_2 + 1))$, so that $k = (\beta - 1)t(\Delta_2 + 1)$. Now

$$\begin{aligned} k^2 - (t-1) \cdot |A^*(u)| &= ((\beta - 1)t(\Delta_2 + 1))^2 - \beta t \Delta_2(\Delta_2 + 1)(t - 1) \\ &= \left((\beta - 1)^2 - \gamma \cdot \beta\right) \cdot (t(\Delta_2 + 1))^2, \end{aligned}$$

which is positive if and only if $(\beta - 1)^2 - \gamma\beta > 0$, which holds true because $\beta \geq \alpha > \alpha^*$. Thus, by Corollary 2.2.3, we obtain

$$|B(u)| \leq |A^*(u)| \cdot \frac{k - (t-1)}{k^2 - (t-1) \cdot |A^*(u)|} = \frac{1 - \frac{t-1}{k}}{\frac{k}{|A^*(u)|} - \frac{t-1}{k}}.$$

The numerator and denominator of the right hand side are both positive, so we can bound and rearrange as follows:

$$\begin{aligned}
|B(u)| &\leq \left(\frac{k}{|A^*(u)|} - \frac{t-1}{k}\right)^{-1} = \left(\frac{(\beta-1)t(\Delta_2+1)}{\beta t\Delta_2(\Delta_2+1)} - \frac{t-1}{(\beta-1)t(\Delta_2+1)}\right)^{-1} \\
&= \Delta_2 \cdot \left(\frac{\beta-1}{\beta} - \frac{1}{\beta-1}\cdot\frac{\Delta_2}{\Delta_2+1}\cdot\frac{t-1}{t}\right)^{-1} = \Delta_2 \cdot \left(\frac{\beta-1}{\beta} - \frac{\gamma}{\beta-1}\right)^{-1} \\
&\leq \Delta_2 \cdot \frac{\alpha(\alpha-1)}{(\alpha-1)^2-\gamma\alpha}, \qquad (2.7)
\end{aligned}$$

where the last step holds because $\beta \geq \alpha > \alpha^*$ and α^* is the larger singular point of $\frac{\beta(\beta-1)}{(\beta-1)^2-\gamma\beta}$, which is a decreasing function of β for all $\beta > \alpha^*$.

Evaluating (2.7) and Claim 2.3.4 in the upper bound of Claim 2.3.32 yields

$$\begin{aligned}
n &\leq |N_1^*(u)| + |N_2(u)| + |B(u)| + |N_2(N_1(u))| \\
&\leq (t-1)\cdot\Delta_2 + \Delta_2 + \frac{\alpha(\alpha-1)}{(\alpha-1)^2-\alpha}\cdot\Delta_2 + \Delta_1\Delta_2 \\
&= \left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2-\alpha}\right)\cdot\Delta_2 + \Delta_1\Delta_2. \qquad (2.8)
\end{aligned}$$

Bound for Case 2. Choose labellings of G_1 and G_2 such that there is a unique purple edge uv that satisfies $|N_2(u) \cap N_2(v)| < (1-\epsilon)\cdot\Delta_2$. Note that the inequalities $|A^*(u)| < \alpha t\cdot\Delta_2(\Delta_2+1)$ and $|A^*(v)| < \alpha t\cdot\Delta_2(\Delta_2+1)$ are satisfied as well, as a direct consequence of the assumptions of Case 2.

We proceed with deriving a technical estimate on an intersection of neighbourhood sets. For each $x \in N_2(u)\setminus N_2(v)$ and $y \in N_2(v)\setminus N_2(u)$ we have $x \neq y$ and therefore absence of blue copies of $K_{2,t}$ implies the inequality $|N_1(x)\cap N_1(y)| \leq t-1$. So

$$\begin{aligned}
|N_1(N_2(u)\setminus N_2(v)) \cap N_1(N_2(v)\setminus N_2(u))| &\leq \sum_{x\in N_2(u)\setminus N_2(v)}\ \sum_{y\in N_2(v)\setminus N_2(u)} |N_1(x)\cap N_1(y)| \\
&\leq |N_2(u)\setminus N_2(v)|\cdot|N_2(v)\setminus N_2(u)|\cdot(t-1) \\
&\leq (\Delta_2 - |N_2(u)\cap N_2(v)|)^2\cdot(t-1).
\end{aligned}$$

Furthermore, since $|N_2(u)\cap N_2(v)| < (1-\epsilon)\cdot\Delta_2$,

$$\begin{aligned}
|N_1(N_2(u)) \cap N_1(N_2(v))| &\leq |N_1(N_2(u)\cap N_2(v))| + |N_1(N_2(u)\setminus N_2(v))\cap N_1(N_2(v)\setminus N_2(u))| \\
&< \Delta_1\cdot|N_2(u)\cap N_2(v)| + (\Delta_2 - |N_2(u)\cap N_2(v)|)^2\cdot(t-1) \\
&\leq \max_{p\in\{0,1,2,\ldots,\lfloor(1-\epsilon)\cdot\Delta_2\rfloor\}} \left(\Delta_1\cdot p + (\Delta_2-p)^2\cdot(t-1)\right).
\end{aligned}$$

See Figure 2.4. Finally, we evaluate this in Claim 2.3.33 to find the following bound on n:

$$\begin{aligned}
n &\leq |A^*(v)| + |A^*(u)| + |\,(N_2(u)\cup N_1(N_2(u)))\cap(N_2(v)\cup N_1(N_2(v)))\,| \\
&\leq |A^*(v)| + |A^*(u)| + |N_2(u)| + |N_2(v)| + |N_1(N_2(u))\cap N_1(N_2(v))| \\
&\leq 2\alpha t\cdot\Delta_2(\Delta_2+1) + 2\Delta_2 + \max_{p\in\{0,1,2,\ldots,\lfloor(1-\epsilon)\cdot\Delta_2\rfloor\}} \left(\Delta_1\cdot p + (\Delta_2-p)^2\cdot(t-1)\right). \qquad (2.9)
\end{aligned}$$

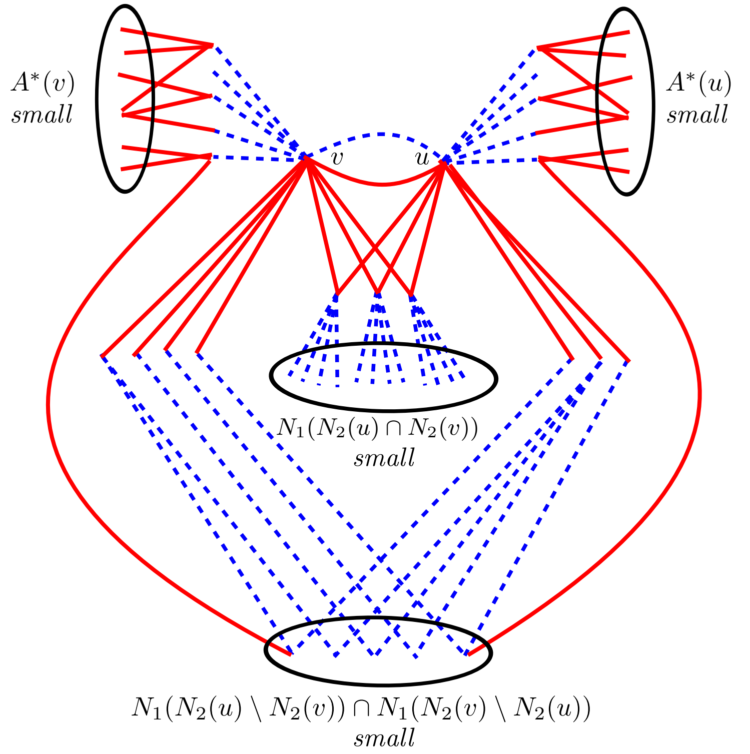

Figure 2.4: A depiction of Case 2 of Theorem 2.1.1, that $|N_1(N_2(u)) \cap N_1(N_2(v))|$ is small.

In particular, this implies the slightly rougher bound

$$n \leq 2\alpha t \cdot \Delta_2(\Delta_2 + 1) + 2\Delta_2 + (1 - \epsilon) \cdot \Delta_1\Delta_2 + \Delta_2^2 \cdot (t - 1). \tag{2.10}$$

Bound for Case 3. Choose a pair of labellings of G_1 and G_2 that induces a unique purple edge uv. The assumptions of this case imply, in particular, that in the red graph the neighbourhoods of each pair of adjacent vertices overlap significantly: $|N_2(x) \cap N_2(y)| \geq (1 - \epsilon) \cdot \Delta_2$ for each $xy \in E(G_2)$.

We will derive two consequences, namely the implication

$$\left(|A^*(u)| \geq 1 + \Delta_2 + \frac{\epsilon \cdot \Delta_2}{1 - 2\epsilon}\right) \implies \left(|B(u)| \leq (t - 1) \cdot \Delta_2^2\right) \tag{2.11}$$

and the inequality

$$|N_2(N_1(u))| \leq \frac{1 + \epsilon}{2}\Delta_1\Delta_2 + \frac{1 - \epsilon}{2}(t - 1) \cdot \Delta_2^2 + \frac{3}{2}\Delta_2. \tag{2.12}$$

We start with proving the statement (2.11), the first consequence. See Figure 2.5. Suppose $a \in A^*(u) \setminus N_2(u)$ has a red neighbour $x \in N_2(u)$. Then ux and ax are edges of G_2, so $|N_2(a) \cap N_2(x)| \geq (1-\epsilon)\Delta_2$ and $|N_2(u) \cap N_2(x)| \geq (1-\epsilon)\Delta_2$. Combining this with the obvious fact that $|N_2(x)| \leq \Delta_2$ yields that

$$|N_2(a) \cap N_2(u)| \geq (1-2\epsilon) \cdot \Delta_2. \tag{2.13}$$

Let us define

$$A^{**}(u) := \{a \in A^*(u) \mid a \text{ has a red neighbour in } N_2(u)\}.$$

It follows from (2.13) that $\sum_{a \in A^{**}(u)} |N_2(a) \cap N_2(u)| \geq |A^{**}(u)| \cdot (1-2\epsilon) \cdot \Delta_2$, so

$$\begin{aligned}\sum_{x \in N_2(u)} |N_2(x)| &\geq \sum_{x \in N_2(u)} |N_2(x) \cap N_2(u)| + \sum_{a \in A^{**}(u)} |N_2(a) \cap N_2(u)| \\ &\geq (1-\epsilon)\Delta_2 \cdot |N_2(u)| + |A^{**}(u)| \cdot (1-2\epsilon) \cdot \Delta_2,\end{aligned}$$

and (crucially) since $\sum_{x \in N_2(u)} |N_2(x)| \leq \Delta_2 \cdot |N_2(u)|$, it follows that

$$|A^{**}(u)| \leq \frac{|N_2(u)| \cdot \Delta_2 - (1-\epsilon) \cdot \Delta_2 |N_2(u)|}{(1-2\epsilon) \cdot \Delta_2} = \frac{\epsilon \cdot |N_2(u)|}{1-2\epsilon}. \tag{2.14}$$

Next, suppose we would have that $|A^*(u)| \geq 1 + |N_2(u)| + |A^{**}(u)|$. Then there exists a vertex $a \in A^*(u) \setminus A^{**}(u)$. By the definition of $A^{**}(u)$, this vertex satisfies $N_2(a) \cap N_2(u) = \emptyset$. Furthermore, since $a \in A^*(u)$, we have that for all $b \in B(u)$ there is a red–blue-link from a to b. In other words, $B(u) = N_1(N_2(a)) \cap B(u)$. This implies that $|B(u)| = |N_1(N_2(a)) \cap B(u)| \leq |N_1(N_2(a)) \cap N_1(N_2(u))| \leq (t-1) \cdot \Delta_2^2$, where the last inequality is a consequence of the facts that $N_2(a) \cap N_2(u) = \emptyset$ and G_1 does not contain a copy of $K_{2,t}$. In summary, we have shown the implication

$$|A^*(u)| \geq 1 + |N_2(u)| + |A^{**}(u)| \implies |B(u)| \leq (t-1) \cdot \Delta_2^2. \tag{2.15}$$

Combining (2.14) and (2.15) yields our first desired main consequence (2.11).

We now prove inequality (2.12), the second consequence. See Figure 2.6. First, the absence of blue copies of $K_{2,t}$ implies that for every $x \in N_2(u)$ we have $|N_1(x) \cap N_1(u)| \leq t-1$. Therefore

$$|N_1(u) \cap N_1(N_2(u))| \leq |N_2(u)| \cdot \max_{x \in N_2(u)} (|N_1(x) \cap N_1(u)|) \leq \Delta_2 \cdot (t-1).$$

In other words, there is a red–blue-link from u to y for at most $\Delta_2 \cdot (t-1)$ vertices $y \in N_1(u)$. Recalling that there is a link from u to every vertex (possibly with the exception of v), it follows that there are at least $h := |N_1(u)| - (t-1)\Delta_2 - 1$ vertices $y \in N_1(u)$ for which there is a blue–red-link (and no red–blue-link) from u to y. In other words, $m := |N_1(u) \cap N_2(N_1(u))| \geq h$. It follows from the definition of blue–red-link that any $y_1 \in N_1(u) \cap N_2(N_1(u))$ is connected to at least one other vertex $y_2 \in N_1(u) \cap N_2(N_1(u))$ by a *red edge*.

This means that $N_1(u) \cap N_2(N_1(u))$ can be covered by a collection of vertex-disjoint red stars $S_1, S_2, \ldots$ that each have at least two vertices (unless $m \in \{0, 1\}$, in which

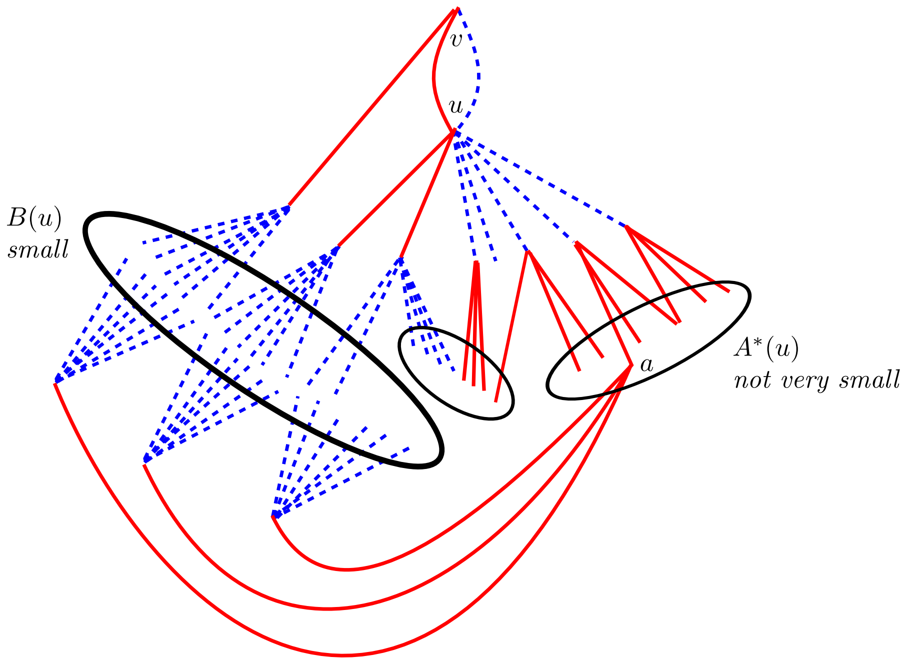

Figure 2.5: A depiction of (2.11) in Case 3 of Theorem 2.1.1.

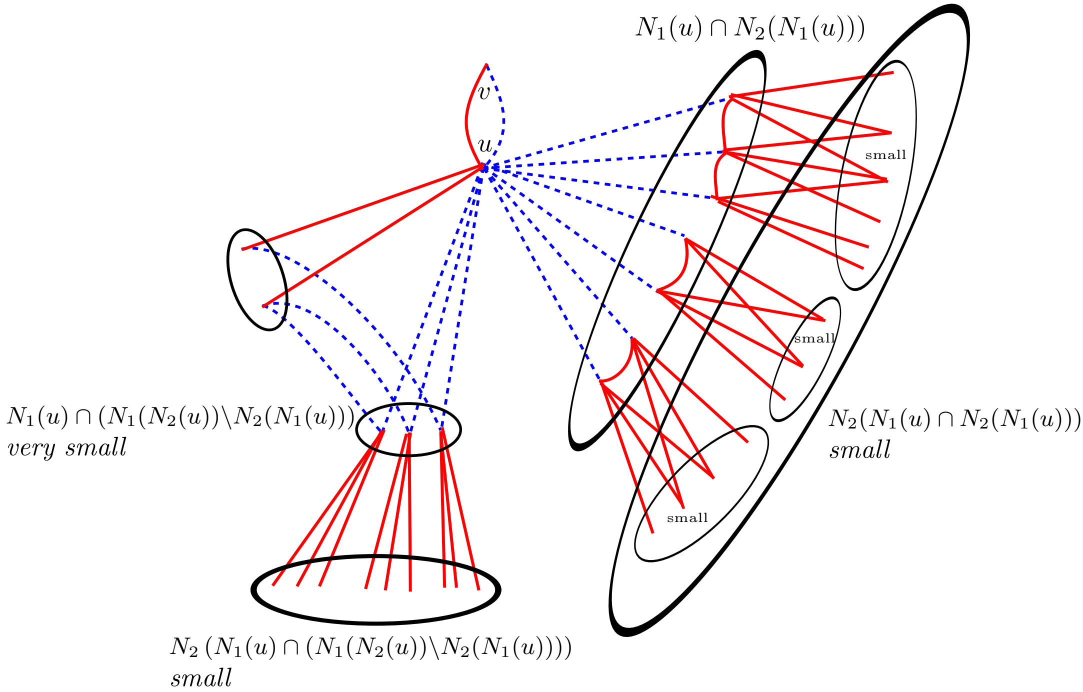

Figure 2.6: A depiction of (2.12) in Case 3 of Theorem 2.1.1.

case inequality (2.12) is clearly satisfied). Let S be one such star, with central vertex y^* and leaves $y_1, y_2 \ldots, y_{|S|-1}$. Each of its edges has a large common red neighbourhood: for all $j \in \{1, 2, \ldots, |S|-1\}$ it holds that $|N_2(y^*) \cap N_2(y_j)| \geq (1-\epsilon) \cdot \Delta_2$. Therefore $|\bigcup_{y \in S} N_2(y)| \leq |N_2(y^*)| + \sum_{y \in S \backslash \{y^*\}} |N_2(y) \backslash N_2(y^*)| \leq (1 + \epsilon \cdot (|S|-1)) \cdot \Delta_2$, which is at most $\frac{1+\epsilon}{2} \cdot |S| \cdot \Delta_2$. So

$$
\begin{aligned}
|N_2\left(N_1(u) \cap N_2(N_1(u))\right)| = \left|\bigcup_i \bigcup_{y \in S_i} N_2(y)\right| &\leq \sum_i \left|\bigcup_{y \in S_i} N_2(y)\right| \\
&\leq \sum_i \frac{1+\epsilon}{2} \cdot |S_i| \cdot \Delta_2 = \frac{m}{2} \cdot (1+\epsilon) \cdot \Delta_2.
\end{aligned}
$$

Last, note that

$$|N_1(u) \cap (N_1(N_2(u)) \setminus N_2(N_1(u)))| = |N_1(u)| - m - \mathbb{1}_{\{\nexists \text{ link from } u \text{ to } v\}} \leq |N_1(u)| - m.$$

We are now ready to derive (2.12):

$$
\begin{aligned}
|N_2(N_1(u))| \quad &\leq \quad |N_2\left(N_1(u) \cap N_2(N_1(u))\right)| + \\
&\qquad |N_2\left(N_1(u) \cap (N_1(N_2(u)) \setminus N_2(N_1(u)))\right)| + |N_2(v)| \\
&\leq \quad \frac{m}{2} \cdot (1+\epsilon) \cdot \Delta_2 + (|N_1(u)| - m) \cdot \Delta_2 + \Delta_2 \quad =: g(m).
\end{aligned}
$$

Since $\Delta_2 \geq 0$ and $\epsilon < 1/2$, the function $g(x)$ is nonincreasing on the whole of $\mathbb{R}$. Since $h \leq m$, it follows that $g(m) \leq g(h)$. So

$$\begin{aligned}|N_2(N_1(u))| &\leq g(|N_1(u)| - (t-1)\Delta_2 - 1) \\ &= \frac{1+\epsilon}{2} \cdot (|N_1(u)| - (t-1) \cdot \Delta_2 - 1) \cdot \Delta_2 + (t-1) \cdot \Delta_2^2 + 2\Delta_2 \\ &\leq \frac{1+\epsilon}{2} \cdot \Delta_1\Delta_2 + \frac{1-\epsilon}{2} \cdot (t-1) \cdot \Delta_2^2 + \frac{3-\epsilon}{2} \cdot \Delta_2,\end{aligned}$$

as desired.

Finally, we evaluate (2.11) and (2.12) in the bounds on n given by Claim 2.3.3, parts 1 and 2, to obtain

$$\begin{aligned}n &\leq \min\left(|N_1(N_2(u))| + |A^*(u)| + |N_2(u)|,\ |N_2(N_1(u))| + |N_2(u)| + |N_1^*(u)| + |B(u)|\right) \\ &\leq \min\left(\Delta_1\Delta_2 + \Delta_2 + |A^*(u)|,\ \frac{1+\epsilon}{2}\Delta_1\Delta_2 + \frac{1-\epsilon}{2}(t-1)\Delta_2^2 + \left(t + \frac{3-\epsilon}{2}\right)\Delta_2 + |B(u)|\right) \\ &= \Delta_1\Delta_2 + \Delta_2 + \min\left(|A^*(u)|,\ |B(u)| + \left(t + \frac{1-\epsilon}{2}\right)\Delta_2 - \frac{1-\epsilon}{2}\left(\Delta_1\Delta_2 - (t-1)\Delta_2^2\right)\right) \\ &\leq \Delta_1\Delta_2 + \Delta_2 + \max\left(1 + \Delta_2 + \frac{\epsilon\Delta_2}{1-2\epsilon},\ \frac{3-\epsilon}{2}(t-1)\Delta_2^2 - \frac{1-\epsilon}{2}\Delta_1\Delta_2 + \left(t + \frac{1-\epsilon}{2}\right)\Delta_2\right),\end{aligned} \tag{2.16}$$

where we employed Claim 2.3.3 in the first line, Claim 2.3.4 and inequality (2.12) in the second line and implication (2.11) in the last line. □

2.4.2 Proof of Theorem 2.1.2

Suppose the theorem is false. Consider a critical counterexample, a pair of non-packable graphs (G_1, G_2) satisfying the constraints of the theorem, such that there is a near-packing with a unique purple edge uv. We distinguish two cases, Cases 1 and 2. From the first we derive the inequality (2.17) and from the second we obtain the inequality (2.18). Together they contradict the condition that $\max((2.5), (2.6)) < n$, thus proving the theorem.

1. $|A^*(u)| \geq \alpha t \cdot \Delta_2(\Delta_2 + 1)$ or $|A^*(v)| \geq \alpha t \cdot \Delta_2(\Delta_2 + 1)$.

Without loss of generality, we assume $|A^*(u)| \geq \alpha t \cdot \Delta_2(\Delta_2 + 1)$. From here the proof is the same as for Case 1 in the proof of Theorem 2.1.1, leading to the same bound,

$$n \leq \left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2. \tag{2.17}$$

2. Case 1 does not hold.

From here we proceed almost exactly as for Case 2 in the proof of Theorem 2.1.1, the difference being that instead of the upper bound $|N_2(u) \cap N_2(v)| < (1-\epsilon) \cdot \Delta_2$ we use $|N_2(u) \cap N_2(v)| < s$, which holds due to the additional condition $\Delta_2^{\triangle} < s$. (Compare with (2.10).) It follows that

$$n \leq 2\alpha t \cdot \Delta_2(\Delta_2 + 1) + 2\Delta_2 + \Delta_1 \cdot (s-1) + \Delta_2^2 \cdot (t-1). \tag{2.18}$$

□

Concluding remarks

We wish to make the following remarks about Theorems 2.1.1 and 2.1.2.

- In Theorem 2.1.1, the bottleneck is the quantity (2.2), which corresponds to the bound (2.10) of Case 2. So improving in this case would improve the overall bound on n, albeit not by much.

- The condition in Theorem 2.1.2 that $\Delta_2^{\triangle} < s$ is equivalent to "$|N_2(x) \cap N_2(y)| < s$ for all $xy \in E(G_2)$". With a little adaptation, we can replace this by the weaker but perhaps obscure condition that G_2 has *no* subgraph $G_2^!$ such that $|N_2(x) \cap N_2(y)| \geq s$ for *all* $xy \in E(G_2^!)$. Indeed, this property is invariant under edge removal, and so holds for an edge-minimal critical counterexample, which therefore has an edge uv with $|N(u) \cap N(v)| < s$, for which we can choose labellings such that uv is the unique purple edge. From here, one again proceeds exactly as in Case 2 of the proof of Theorem 2.1.1.

- Theorem 2.1.2 yields a better bound than Theorem 2.1.1 only if Δ_1 is much larger than Δ_2 and s, t are both small.

2.5 Application to packing graphs of even girth 10

The purpose of this section is to provide an application of Theorem 2.1.1. We give a combinatorial proof that the BEC conjecture holds for every pair of graphs neither of which contains a 4-, 6- or 8-cycle as a subgraph — i.e. both of which have even girth at least 10 — provided at least one of the graphs has large enough maximum degree. From now on, we always assume for convenience that $\Delta_1 \geq \Delta_2$.

Theorem 2.5.1. *If G_1 and G_2 are graphs on n vertices with respective maximum degrees Δ_1 and Δ_2 such that $(\Delta_1+1)(\Delta_2+1) \leq n+1$, then they pack provided neither contains a 4-, 6- or 8-cycle and either $\Delta_1 \geq 940060$ or $\Delta_1 \geq \Delta_2 \geq 27620$.*

Central to the proof of Theorem 2.1.1 was a lemma of Corrádi [30]. We use it again for the proof of Theorem 2.5.1, but the application is slightly more involved as we shall see in Section 2.5.2.

It is important to note that Theorem 2.5.1 can also be obtained by applying the earlier work of Bollobás, Kostochka and Nakprasit [15, Thm. 2] for packing with a d-degenerate graph, provided we also use a classic bound on the Turán number of even cycles [16], cf. also [90]. The novel contribution here is thus our proof strategy, which in particular does not use any probabilistic methodology. We have optimism that this strategy may be helpful for other related graph packing problems.

As mentioned, an important ingredient of the proof of Theorem 2.5.1 is the following special case ($t = 2$) of Corollary 2.1.5.

Corollary 2.5.2. *If G_1 and G_2 are graphs on n vertices with respective maximum degrees Δ_1 and Δ_2 such that $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$, then they pack provided G_1 contains no 4-cycle and $\Delta_1 > 34\Delta_2$.*

Therefore, for Theorem 2.5.1, we may restrict our attention to the case where Δ_1 and Δ_2 are relatively close to each other, i.e. $\Delta_2 \leq \Delta_1 \leq 34\Delta_2$. The following theorem yields a bound that is sufficiently strong in that regime, and so it is the main objective of this paper.

Theorem 2.5.3. *Two graphs G_1 and G_2 on n vertices with respective maximum degrees $\Delta_1 \geq \Delta_2$ pack if the following two properties hold.*

- *Neither contains a 4-, 6- or 8-cycle*
- *There exists an integer $t \geq 2$ for which*

$$4C_t \cdot \Delta_1\sqrt{\Delta_1} + \tfrac{4}{t} \cdot \Delta_1\Delta_2 + 7(\Delta_1 + \Delta_2) < n,$$

where $C_t := \frac{\sqrt{1.37}}{0.37\sqrt{t-1}} + \sqrt{1.37(t-1)}$.

Let us now briefly show how Theorems 2.5.2 and 2.5.3 imply Theorem 2.5.1.

Proof of Theorem 2.5.1. Suppose there are graphs G_1 and G_2 that form a counterexample. Routine arithmetic manipulations show that, if for some $t \geq 2$

$$\sqrt{\Delta_1} < \frac{t-4}{4tC_t}\Delta_2 - \frac{3}{C_t} = \frac{1}{4tC_t}((t-4)\Delta_2 - 12t), \tag{2.19}$$

then $4C_t \cdot \Delta_1\sqrt{\Delta_1} + \frac{4}{t} \cdot \Delta_1\Delta_2 + 7(\Delta_1 + \Delta_2)$ is strictly less than $(\Delta_1 + 1)(\Delta_2 + 1) - (1 + 6(\Delta_1 - \Delta_2)) \leq (\Delta_1 + 1)(\Delta_2 + 1) - 1 \leq n$. So G_1 and G_2 pack by Theorem 2.5.3, contradiction.

Moreover, by Theorem 2.5.2, if

$$\Delta_1 \geq 34 \cdot \Delta_2, \tag{2.20}$$

then G_1 and G_2 pack, also a contradiction. Thus neither of (2.19) and (2.20) holds, and so

$$\frac{136tC_t}{t-4}\left(\sqrt{\Delta_1} + \frac{3}{C_t}\right) \geq 34\Delta_2 > \Delta_1 \geq \frac{1}{16t^2C_t^2}\left((t-4)\Delta_2 - 12t\right)^2.$$

This in turn yields the following two quadratic polynomial inequalities:

$$(t-4)^2\Delta_2^2 - (544t^2C_t^2 + 24t)\Delta_2 + 144t^2 < 0 \text{ and}$$
$$(t-4)\Delta_1 - 136tC_t\sqrt{\Delta_1} - 408t < 0.$$

A good choice of t turns out to be $t = 15$. Substituting this (and the formula for C_t) into the above two inequalities yields that $\Delta_2 < 27620$ and $\Delta_1 < 940060$. This contradicts our assumptions on Δ_1 and Δ_2, and this completes the proof. □

We have made little effort to optimise the boundary constants 940060 and 27620. These constants partly depend on the constant 34 in Corallary 2.5.2, which we believe can be lowered. To be more specific, any improvement of the constant 34 by a factor C will yield a corresponding improvement by approximately a factor C^2 for the bounds on Δ_1 and Δ_2 in Theorem 2.5.1.

2.5.1 A hypothetical critical counterexample to Theorem 2.5.1

We begin the proof of Theorem 2.5.3 in this section and continue it in the next two sections. Our proof is by contradiction. Just as in the proofs of Theorems 2.1.1 and 2.1.2, we consider an edge-minimal counterexeample. That is, we choose a pair (G_1, G_2) of non-packable graphs on n vertices that satisfy the conditions of Theorem 2.5.1, where G_2 is edge-minimal over all such pairs (G_1, G_2). This again yields a labelling of G_1 and G_2 such that $e = uv \in E(G_2)$ is the unique purple edge. Thus, we can define $A(u), B(u), A^*(u)$ and $B^*(u)$ as before. These sets are analogously defined for v also, and indeed for any element of $[n]$. The following two claims are analogues of claims 2.3.1 and 2.3.2, with exactly the same proof.

Claim 2.5.4. *For all $w \in [n] \setminus \{v\}$, there is a red–blue-link or a blue–red-link from u to w.*

For all $w \in [n] \setminus \{u\}$, there is a red–blue-link or a blue–red-link from v to w.

Claim 2.5.5. *For all $a \in A^*(u)$ and $b \in B(u)$, there is a red–blue-link from a to b. For all $b \in B^*(u)$ and $a \in A(u)$, there is a blue–red-link from b to a.*

We may assume that $\Delta_1, \Delta_2 \geq 2$ since the BEC conjecture is known for $\Delta_2 = 1$. Then the following claim shows that neither of $A^*(u)$ and $B^*(u)$ is empty.

Claim 2.5.6. $|A^*(u)| \geq \Delta_1 - 1$ *and* $|B^*(u)| \geq \Delta_2 - 1$*. And so* $|A^*(u)|, |B^*(u)| \geq 1$.

Proof. Suppose otherwise. If $|A^*(u)| \leq \Delta_1 - 2$, note that $[n] \subseteq N_1(N_2(u)) \cup A^*(u) \cup N_2(u)$ by Claim 2.5.4, and so

$$n \leq |N_1(N_2(u))| + |A^*(u)| + |N_2(u)| \leq \Delta_1\Delta_2 + \Delta_1 - 2 + \Delta_2.$$

Symmetrically, if $|B^*(u)| \leq \Delta_2 - 2$, then

$$n \leq |N_2(N_1(u))| + |B^*(u)| + |N_1(u)| \leq \Delta_1\Delta_2 + \Delta_2 - 2 + \Delta_1.$$

In either case, we obtain a contradiction to the assumption that $n \geq (\Delta_1 + 1)(\Delta_2 + 1) - 1$. □

2.5.2 Bounding second order neighbourhoods

The following technical bound forms the core of the argument. It bounds the intersection of any two mixed second order neighbourhoods in our hypothetical critical counterexample. The bound relies on an application of Corrádi's lemma (Lemma 2.2.3).

Claim 2.5.7. *For any integer $t \geq 2$ and distinct $a, b \in [n]$,*

$$|N_1(N_2(a)) \cap N_1(N_2(b))| \leq \Delta_1 + \Delta_2 + \sqrt{1.37(t-1)}\Delta_2\sqrt{\Delta_2} + \frac{\sqrt{1.37}}{0.37\sqrt{t-1}}\Delta_1\sqrt{\Delta_2} + \frac{1}{t}\Delta_1\Delta_2 \text{ and}$$

$$|N_2(N_1(a)) \cap N_2(N_1(b))| \leq \Delta_1 + \Delta_2 + \sqrt{1.37(t-1)}\Delta_1\sqrt{\Delta_1} + \frac{\sqrt{1.37}}{0.37\sqrt{t-1}}\Delta_2\sqrt{\Delta_1} + \frac{1}{t}\Delta_1\Delta_2.$$

Proof. By symmetry we only need to prove the first bound. Our approach to this is to partition $N_1(N_2(a)) \cap N_1(N_2(b))$ into a number of subsets, each of which we bound separately. To assist the reader, we have provided a depiction of our partition scheme in Figure 2.7.

Before starting the main argument, we first need to prune the neighbourhood $N_1(N_2(a))$ of three types of relatively small subsets.

- First, since G_2 is C_4-free, $|N_2(a) \cap N_2(b)| \leq 1$, so

$$|N_1(N_2(a) \cap N_2(b))| \leq \Delta_1. \tag{2.21}$$

  Thus we can restrict our attention to $|N_1(N_2(a) \setminus N_2(b)) \cap N_1(N_2(b))|$. The reason for this technical reduction is so that we can work with the *disjoint* sets $N_2(a) \setminus N_2(b)$ and $N_2(b)$.

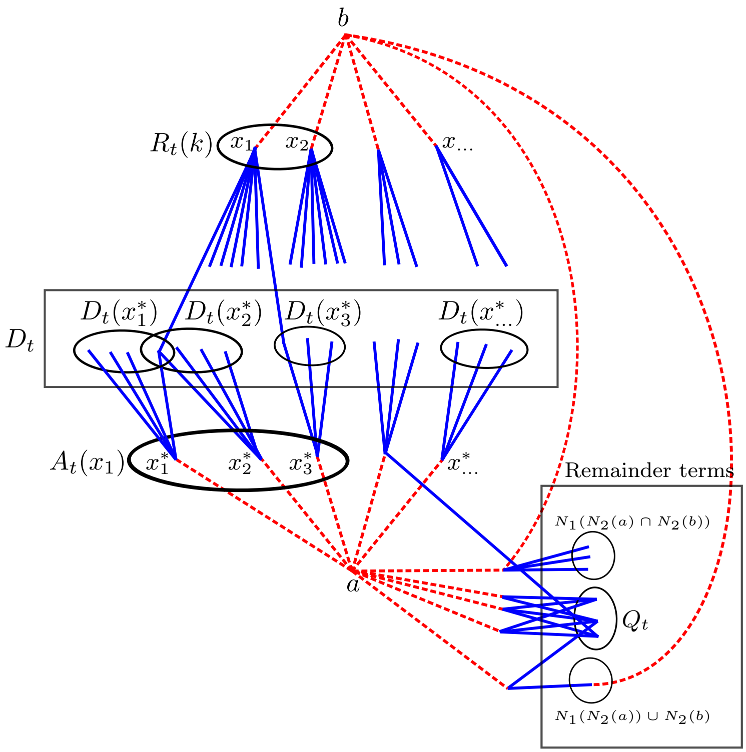

Figure 2.7: A depiction of the vertex sets relevant to the proof of Claim 2.5.7.

- Second, define

$$Q_t := \{y \in N_1(N_2(a)) \mid |N_1(y) \cap N_2(a)| \geq t\}.$$

So Q_t is the set of vertices in $N_1(N_2(a))$ that are in the blue neighbourhoods of at least t different red neighbours of a. We estimate $|Q_t|$ separately, because its elements facilitate a large amount of overlap among the blue neighbourhoods of (at most t different) vertices in $N_2(a)$, while still not violating the absence of large cycles. By an overcounting argument,

$$|Q_t| \leq \sum_{x \in N_2(a)} \sum_{y \in N_1(x)} \frac{\mathbb{1}_{\{y \in Q_t\}}}{t} \leq \frac{1}{t} \sum_{x \in N_2(a)} \sum_{y \in N_1(x)} 1 \leq \frac{\Delta_1 \Delta_2}{t}. \tag{2.22}$$

- Third, we estimate $|N_1(N_2(a)) \cap N_2(b)|$ separately, because later we wish to be able to assume that there are no blue edges between $N_2(a)$ and $N_2(b)$. We have that

$$|N_1(N_2(a)) \cap N_2(b)| \leq |N_2(b)| \leq \Delta_2. \tag{2.23}$$

Having established the estimates (2.21), (2.22) and (2.23) separately, we are left with estimating $|N_1(N_2(b)) \cap (N_1(N_2(a) \setminus N_2(b)) \setminus (Q_t \cup N_2(b)))|$, and we do so with Lemma 2.2.3.

For brevity, define $D_t := N_1(N_2(a) \setminus N_2(b)) \setminus (Q_t \cup N_2(b))$ and $D_t(x^*) := N_1(x^*) \setminus (Q_t \cup N_2(b))$ for any vertex $x^* \in N_2(a) \setminus N_2(b)$. Note that $D_t = \bigcup_{x^* \in N_2(a) \setminus N_2(b)} D_t(x^*)$ and our goal now is to bound $|N_1(N_2(b)) \cap D_t|$.

Define $k := \sqrt{1.37(t-1)\Delta_2}$ and let

$$R_t(k) := \{x \in N_2(b) \mid |N_1(x) \cap D_t| > k\}.$$

So $R_t(k)$ is the set of red neighbours of b that each have 'large' blue neighbourhoods intersecting D_t. We want to show that $|R_t(k)|$ is small, so without loss of generality we may assume that k is small enough to ensure that $R_t(k) \neq \emptyset$.

For each $x \in N_2(b)$, define the set

$$A_t(x) := \{x^* \in N_2(a) \setminus N_2(b) \mid N_1(x) \cap D_t(x^*) \neq \emptyset\}.$$

For the moment, let us assume that we have established the following two properties:

$$|A_t(x)| > k \qquad \text{for all } x \in R_t(k); \tag{2.24}$$
$$|A_t(x_1) \cap A_t(x_2)| \leq t-1 \qquad \text{for all distinct } x_1, x_2 \in N_2(b). \tag{2.25}$$

We prove these two properties later, but let us first show how from these both a bound on $|R_t(k)|$ and then the desired result follow.

Note that we have chosen k such that $k^2 = 1.37(t-1)\Delta_2 > (t-1)|N_2(a) \setminus N_2(b)|$. By this choice and the inequalities in (2.24) and (2.25), we may apply Lemma 2.2.3

with $N = |R_t(k)|$, $X = N_2(a) \setminus N_2(b)$, the parameters t and k, and the collection $(A_t(x))_{x\in R_t(k)}$ of subsets of X, yielding the following bound:

$$\begin{aligned}|R_t(k)| &\le |N_2(a)\setminus N_2(b)|\cdot \frac{k-(t-1)}{k^2-(t-1)|N_2(a)\setminus N_2(b)|}\\ &\le \Delta_2\cdot\frac{\sqrt{1.37(t-1)\Delta_2}}{1.37(t-1)\Delta_2-(t-1)\Delta_2} = \frac{\sqrt{1.37}}{0.37}\sqrt{\frac{\Delta_2}{t-1}}.\end{aligned}$$

We can then bound the main term as follows:

$$\begin{aligned}|N_1(N_2(b))\cap D_t| &\le |\{x\in N_2(b) \mid |N_1(x)\cap D_t|\le k\}|\cdot k + |R_t(k)|\Delta_1\\ &\le \Delta_2 k + |R_t(k)|\Delta_1 \le \Delta_2\sqrt{1.37(t-1)\Delta_2} + \frac{\sqrt{1.37}}{0.37}\sqrt{\frac{\Delta_2}{t-1}}\Delta_1\\ &= \frac{\sqrt{1.37}}{0.37\sqrt{t-1}}\Delta_1\sqrt{\Delta_2} + \sqrt{1.37(t-1)}\Delta_2\sqrt{\Delta_2}. \end{aligned}\tag{2.26}$$

Combining inequalities (2.21), (2.22), (2.23) and (2.26), we obtain

$$\begin{aligned}&|N_1(N_2(b))\cap N_1(N_2(a))|\\ &\le |N_1(N_2(b))\cap D_t| + |N_1(N_2(a)\cap N_2(b))| + |N_1(N_2(b))\cap Q_t| + |N_1(N_2(b))\cap N_2(b)|\\ &\le \frac{\sqrt{1.37}}{0.37\sqrt{t-1}}\Delta_1\sqrt{\Delta_2} + \sqrt{1.37(t-1)}\Delta_2\sqrt{\Delta_2} + \Delta_1 + \frac{1}{t}\Delta_1\Delta_2 + \Delta_2,\end{aligned}$$

which is the desired result.

So to complete the proof, it only remains to show the two properties (2.24) and (2.25).

For (2.24), since G_1 has no 4-cycle, it holds that $|N_1(x)\cap D_t(x^*)| \le 1$ for each $x \in N_2(b)$ and $x^* \in N_2(a)\setminus N_2(b)$. So for a fixed $x \in R_t(k) \subseteq N_2(b)$, each $x^* \in N_2(a)\setminus N_2(b)$ contributes at most 1 to $|N_1(x)\cap D_t|$. This proves (2.24).

To prove (2.25), suppose for a contradiction that there exist distinct $x_1, x_2 \in N_2(b)$ such that $|A_t(x_1)\cap A_t(x_2)| \ge t$. Then there are at least t different vertices $x_1^*, \dots, x_t^* \in N_2(a)\setminus N_2(b)$, and there exist vertices $y_{11} \in D_t(x_1^*)\cap N_1(x_1), \dots, y_{t1} \in D_t(x_t^*)\cap N_1(x_1)$ as well as vertices $y_{12} \in D_t(x_1^*)\cap N_1(x_2), \dots, y_{t2} \in D_t(x_t^*)\cap N_1(x_2)$. Due to the separate estimate (2.22), we were allowed to exclude elements of the set Q_t in our choice of the sets $D_t(\cdot)$, and so the vertices $y_{11}, \dots y_{t1}$ are not all equal. Recall that we assumed $t \ge 2$. Without loss of generality, we may assume that $y_{11} \ne y_{21}$. Note though that some of the vertices y_{11}, y_{21}, y_{12}, y_{22} may well be equal. Due to the separate estimate (2.23), we were also allowed to exclude elements of $N_2(b)$ in our choice of $D_t(\cdot)$, and so $x_1x_1^*$, $x_1x_2^*$, $x_2x_1^*$, $x_2x_2^*$ are not blue edges. Therefore $\{x_1, x_2, x_1^*, x_2^*\}\cap\{y_{11}, y_{12}, y_{21}, y_{22}\} = \emptyset$.

It can be shown that the induced subgraph $G_1[\{x_1, x_2, x_1^*, x_2^*, y_{11}, y_{12}, y_{21}, y_{22}\}]$ contains a 4-, 6- or 8-cycle, which is a contradiction. To wit, the case analysis proceeds as follows. See Figure 2.8 for a pictorial synopsis. Since $y_{11} \ne y_{21}$, there are four cases for the possible coincidences among y_{11}, y_{21}, y_{12}, y_{22}:

1. *The vertices are all distinct.* Then $y_{11}x_1y_{21}x_2^*y_{22}x_2y_{12}x_1^*$ is a blue 8-cycle.
2. *Exactly one pair of the vertices coincides.* Since $y_{11} \ne y_{21}$, there are five subcases: $y_{11} = y_{12}$, $y_{11} = y_{22}$, $y_{12} = y_{21}$, $y_{12} = y_{22}$, or $y_{21} = y_{22}$. We can consider each

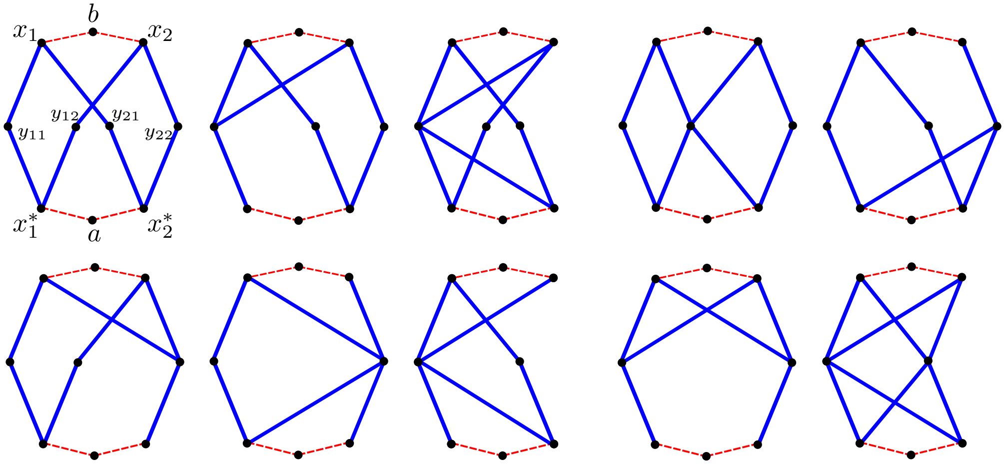

Figure 2.8: The cases analysed in Claim 2.5.7. We know $y_{11} \neq y_{21}$, but some of the vertices y_{11}, y_{12}, y_{21}, y_{22} may coincide. As shown here, in each case there is a blue 4-, 6- or 8-cycle. In reading order, the depicted cases are: (a) all are distinct, (b)–(f) exactly one pair of vertices coincides, (g)–(h) a triple of vertices coincides, (i)–(j) two pairs of vertices coincide.

subcase individually (as in Figure 2.8), or we can also notice some symmetries by a relabelling of the vertices $x_1, x_2, x_1^*, x_2^*, y_{11}, y_{12}, y_{21}, y_{22}$. The three subcases $y_{11} = y_{12}$, $y_{12} = y_{22}$ and $y_{21} = y_{22}$ are symmetric, and in the first of these subcases $y_{11}x_1y_{21}x_2^*y_{22}x_2$ is a blue 6-cycle. The two remaining subcases $y_{11} = y_{22}$ and $y_{12} = y_{21}$ are symmetric, and in the first of these $y_{11}x_1y_{21}x_2^*$ is a blue 4-cycle.

3. *A triple of the vertices coincides.* Since $y_{11} \neq y_{21}$, there are two subcases: $y_{12} = y_{21} = y_{22}$ or $y_{11} = y_{12} = y_{22}$. In the first of these $y_{11}x_1y_{12}x_1^*$ is a blue 4-cycle, while in the second $y_{11}x_1y_{21}x_2^*$ is a blue 4-cycle.

4. *Two pairs of the vertices coincide.* Since $y_{11} \neq y_{21}$, there are two subcases: $y_{11} = y_{12}, y_{21} = y_{22}$ or $y_{11} = y_{22}, y_{12} = y_{21}$. In both of these $y_{11}x_1y_{21}x_2$ is a blue 4-cycle. □

We in fact use a weaker but handier version of Claim 2.5.7. For each $t \geq 2$, define

$$C_t := \frac{\sqrt{1.37}}{0.37\sqrt{t-1}} + \sqrt{1.37(t-1)}.$$

Claim 2.5.8. *For each $t \geq 2$, we have that $\Delta_1 + \Delta_2 + C_t\Delta_1\sqrt{\Delta_1} + \Delta_1\Delta_2/t$ is an upper bound for each of the following quantities: $|N_1(N_2(u)) \cap N_1(N_2(v))|$, $|N_2(N_1(u)) \cap N_2(N_1(v))|$, $|A(v)|$, $|B(v)|$, $|A(u)|$, $|B(u)|$.*

Proof. For the first two quantities, apply Claim 2.5.7 with $a = v$ and $b = u$ and note that $\Delta_1 \geq \Delta_2$, by assumption.

For the last quantity, note first that $|A^*(u)| \geq 1$ by Claim 2.5.6. By Claim 2.5.5, there exists $a \in A^*(u)$ (not equal to u) such that $B(u) \subseteq N_1(N_2(a)) \cap N_1(N_2(u))$. The bound follows from Claim 2.5.7 with a and $b = u$ and the assumption that $\Delta_1 \geq \Delta_2$. The proof for the remaining quantities is the same. □

2.5.3 Putting the neighbourhood bounds together

We are ready to complete the proof of Theorem 2.5.3.

By Claim 2.5.4 we have that

$$\begin{aligned}
[n] &\subseteq N_1(N_2(u)) \cup A^*(u) \cup N_2(u), \\
[n] &\subseteq N_1(N_2(v)) \cup A^*(v) \cup N_2(v), \text{ and} \\
[n] &\subseteq N_2(N_1(v)) \cup B^*(v) \cup N_1(v).
\end{aligned}$$

So it follows (also using the definitions of $A^*(v)$, $A(v)$, $A^*(u)$, $B^*(v)$, $A(u)$, $B(v)$) that

$$\begin{aligned}
n &\leq |N_1(N_2(u))| + |A^*(u)| + |N_2(u)| \\
&\leq (|N_1(N_2(u)) \cap N_1(N_2(v))| + |N_1(N_2(u)) \cap A^*(v)| + |N_1(N_2(u)) \cap N_2(v)|) + \\
&\qquad (|A^*(u) \cap N_2(N_1(v))| + |A^*(u) \cap B^*(v)| + |A^*(u) \cap N_1(v)|) + |N_2(u)| \\
&\leq (|N_1(N_2(u)) \cap N_1(N_2(v))| + |A(v)| + |N_1(v)| + |N_2(v)|) + \\
&\qquad (|N_2(N_1(u)) \cap N_2(N_1(v))| + |A(u) \cap B(v)| + |N_1(u)| + |N_2(v)| + |N_1(v)|) + |N_2(u)| \\
&\leq |N_1(N_2(u)) \cap N_1(N_2(v))| + |N_2(N_1(u)) \cap N_2(N_1(v))| + |A(v)| + |B(v)| + 3(\Delta_1 + \Delta_2) \\
&\leq 4C_t \Delta_1 \sqrt{\Delta_1} + 4\Delta_1\Delta_2/t + 7(\Delta_1 + \Delta_2), \qquad (2.27)
\end{aligned}$$

where to derive the last line we applied Claim 2.5.8 for some $t \geq 2$. This contradicts our assumption on n. □

2.6 A near-packing interpolation result

A *near-packing of degree d* of a *blue* graph G_1 and a *red* graph G_2 is a pair of injective mappings (labellings) of their vertex sets into $\{1, \ldots, n\}$ such that the graph induced by the intersection of their edge sets (the *purple edges*) has maximum degree d. Note that G_1 and G_2 pack iff they admit a near-packing of degree 0.

In 2000, Eaton [35] gave a short and elegant double-counting proof of a near-packing result that (deceivingly) appears to be close to resolving the BEC-conjecture. She proved that if two graphs G_1, G_2 satisfy the BEC-condition, then they admit a degree ≤ 1 near-packing. Upon inspection of the proof it turns out that she actually establishes something slightly stronger. Indeed, she shows that for *any* labelling of G_1 and G_2 that is not a degree ≤ 1 near-packing, there exist two vertices such that swapping their red labels results in another pair of labellings with fewer purple edges. Thus the following theorem holds. (For completeness, we include the proof, in the language of blue–red–links and red–blue–links.)

Theorem 2.6.1 (Eaton [35]). *Let G_1 and G_2 be graphs that satisfy*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1.$$

Then for any pair of labellings of G_1 and G_2 with fewest purple edges, the graph induced by the purple edges has maximum degree at most 1. Moreover, the number of purple edges is at most $\frac{n}{2} + 1 - \Delta_1 - \Delta_2$.

Proof. Fix a pair of labellings of G_1 and G_2 with fewest purple edges and suppose that the graph induced by the purple edges has maximum degree $d \geq 2$. We will show that then there exist two vertices $i, j \in V(G_2)$ such that a $(i,j)-$swap of the red labels results in a pair of labellings with fewer purple edges; contradiction. For any vertex i, let $\deg_p(i)$ denote the degree of i in the graph induced by the purple edges. From now on, let j be a fixed vertex of maximal purple degree d and let D denote the set of purple neighbours of j. For any other vertex i, we define $\text{Links}(i,j) := \#\{\text{red–blue–links from } i \text{ to } j\} + \#\{\text{blue–red–links from } i \text{ to } j\}$.

Case 1. If $i \notin D, i \neq j$ then

$$\deg_p(i) + \deg_p(j) \leq \text{Links}(i,j).$$

Indeed, if not then an $(i,j)-$swap of the red labels would eliminate $\deg_p(i)+\deg_p(j)$ purple edges, while creating $\text{Links}(i,j)$ new purple edges, thus effectively reducing the number of purple edges; contradiction.

Case 2. If $i \in D$ then

$$\deg_p(i) + \deg_p(j) - 2 \leq \text{Links}(i,j).$$

This is again because an $(i,j)-$swap would reduce the number of purple edges. The extra factor 2 arises because an $(i,j)-$swap does not eliminate the purple edge ij.

Summing the inequalities over all $i \neq j$ yields

$$(n-1) \cdot \deg_p(j) - 2d + \sum_{i \neq j} \deg_p(i) \leq \sum_{i \neq j} \text{Links}(i,j). \tag{2.28}$$

Since $\deg_p(j) = d$ and $\sum_{i \in D} \deg_p(i) \geq d$ and $\sum_{i \neq j} \text{Links}(i,j) \leq 2\Delta_1\Delta_2 - 2d$, it follows that

$$n \cdot d + \sum_{i \notin D} \deg_p(i) \leq 2\Delta_1\Delta_2.$$

This leads to a contradiction if $d \geq 2$, because it implies that $n \leq \Delta_1\Delta_2$. Therefore we must have $d \leq 1$, as desired.

It remains to upperbound the number of purple edges, which equals $\frac{1}{2}\left(\sum_{i \in V(G_2)} \deg_p(i)\right) = \frac{1}{2}\left(1 + \sum_{i \neq j} \deg_p(i)\right)$. By equation (2.28), this is upperbounded by $\frac{1}{2}\left(1 - (n-3)d + \sum_{i \neq j} \text{Links}(i,j)\right) \leq \frac{1}{2}\left(1 - (n-3) + 2\Delta_1\Delta_2 - 2\right) = 1 + \Delta_1\Delta_2 - \frac{n}{2} \leq \frac{n}{2} + 1 - \Delta_1 - \Delta_2$.

□

On the other hand, Sauer and Spencer [94] already showed in 1978 that $2\Delta_1\Delta_2 < n$ is sufficient for G_1 and G_2 to pack. Using the terminology in this chapter, their argument boils down to the fact that in a minimal counterexample, there is a unique purple edge uv such that for every vertex $x \neq v$, there is a link between u and x. Indeed, this implies the contradictory upperbound $n \leq |N_1(N_2(u)) \cup N_2(N_1(u))| \leq 2\Delta_1\Delta_2$.

To conclude this section, we wish to provide a modest new result that interpolates 'linearly' between the bounds of Eaton and Sauer and Spencer.

Lemma 2.6.2 (Interpolation). *Let* $q \in \{0, 1, \ldots, \lfloor n/2 + 1 - \Delta_1 - \Delta_2 \rfloor\}$. *Let* G_1 *be a blue graph with maximum degree* Δ_1 *and let* G_2 *be a red graph with maximum degree* Δ_2. *Supose*

$$\Delta_1 \Delta_2 < \tfrac{n}{2} + q.$$

Then there exists a degree ≤ 1 *near-packing of* G_1 *and* G_2 *with* $\leq q$ *purple edges.*

Proof. Suppose the theorem does not hold, then there exists a nonnegative integer $q \leq \lfloor n/2 + 1 - \Delta_1 - \Delta_2 \rfloor$ and a critical pair of graphs (G_1, G_2) such that $\Delta_1 \Delta_2 < \frac{n}{2} + q$, but every degree 1 near-packing of G_1 and G_2 (if it exists) has $\geq q + 1$ purple edges. Now fix a near-packing of G_1 and G_2 with the minimal possible number z of purple edges. Noting that $\Delta_1 \Delta_2 < \frac{n}{2} + q$ implies $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$, it follows from Theorem 2.6.1 that it is a degree ≤ 1 near-packing. Therefore $z \geq q + 1$.

Let $u_1, u_2, \ldots u_{2z}$ denote the $2z$ distinct vertices that are incident to the z purple edges. Let u_2 be the vertex that is incident to the same purple edge as u_1.

Note that for every $i \neq 2$, the number of blue–red-links from u_1 to u_i plus the number of red–blue-links from u_1 to u_i must be at least 2. For $i = 1$ this follows from the fact that $u_1 u_2$ is a purple edge. For $i \notin \{1, 2\}$ it holds because otherwise it would be possible to swap the red labels of u_1 and u_i, destroying the two different purple edges incident to u_1 respectively u_i while creating at most one new purple edge. This reduces the number of purple edges, contradicting the minimality of z.

Let $C(u_1) := \{x \in V \mid u_1 \text{ and } x \text{ are connected by at least two different links}\}$.

It follows that $|C(u_1)| \geq 2z - 1 \geq 2q + 1$. By a straightforward analogue of Claim 2.3.1, every vertex except possibly u_2 is connected to u_1 by at least one link. Therefore, we can now estimate:

$$n \leq |\{u_2\}| + |N_1(N_2(u_1))| + |N_2(N_1(u_1))| - |C(u_1)| \leq 2\Delta_1 \Delta_2 - 2q.$$

So we obtain the contradictory inequality

$$\Delta_1 \Delta_2 \geq \frac{n}{2} + q.$$

□

Chapter 3

Colouring the square of a claw-free graph

Let G be a claw-free graph on n vertices with clique number ω, and consider the chromatic number $\chi(G^2)$ of the square G^2 of G. Writing $\chi'_s(d)$ for the supremum of $\chi(L^2)$ over the line graphs L of simple graphs of maximum degree at most d, we prove that $\chi(G^2) \leq \chi'_s(\omega)$ for $\omega \in \{3, 4\}$. For $\omega = 3$, this implies the sharp bound $\chi(G^2) \leq 10$. For $\omega = 4$, this implies $\chi(G^2) \leq 22$, which is within 2 of the conjectured best bound. This work is motivated by a strengthened form of a conjecture of Erdős and Nešetřil.

3.1 Introduction

Let G be a claw-free graph, that is, a graph without the complete bipartite graph $K_{1,3}$ as an induced subgraph. We consider the square G^2 of G, formed from G by the addition of edges between those pairs of vertices connected by some two-edge path in G. We seek to optimise the chromatic number $\chi(G^2)$ of G^2 with respect to the clique number $\omega(G)$. We focus on claw-free graphs G having small $\omega(G)$.

The second author with de Joannis de Verclos and Pastor [63] recently conjectured the following. As the class of claw-free graphs is richer than the class of line graphs (cf. e.g. [27]), this is a significant strengthening of a notorious conjecture of Erdős and Nešetřil (cf. [39]).

Conjecture 3.1.1 (de Joannis de Verclos, Kang and Pastor [63])**.** *For any claw-free graph G, $\chi(G^2) \leq \frac{1}{4}(5\omega(G)^2 - 2\omega(G) + 1)$ if $\omega(G)$ is odd, and $\chi(G^2) \leq \frac{5}{4}\omega(G)^2$ otherwise.*

If true, this would be sharp by the consideration of a suitable blow-up of a five-vertex cycle and taking G to be its line graph. The conjecture of Erdős and Nešetřil is the special case in Conjecture 3.1.1 of G the line graph of a (simple) graph. To support the more general assertion and at the same time extend a notable result of Molloy and Reed [86], it was proved in [63] that there is an absolute constant $\varepsilon > 0$ such that $\chi(G^2) \leq (2 - \varepsilon)\omega(G)^2$ for any claw-free graph G.

In this note, our primary goal is to supply additional evidence for Conjecture 3.1.1 when $\omega(G)$ is small. We affirm it for $\omega(G) = 3$ and come to within 2 of the conjectured value when $\omega(G) = 4$. Note that Conjecture 3.1.1 is trivially true when $\omega(G) \leq 2$.

We write $\chi'_s(\omega)$ for the supremum of $\chi(L^2)$ over the line graphs L of all simple graphs of maximum degree ω. Moreover, $\chi'_{s,m}(\omega)$ denotes the supremum of $\chi(L^2)$ over the line graphs L of all multigraphs of maximum degree ω.

Theorem 3.1.2. *Let G be a claw-free graph.*

1. *If $\omega(G) = 3$, then $\chi(G^2) \leq 10$.*
2. *If $\omega(G) = 4$, then $\chi(G^2) \leq 22$; moreover, $\chi(G^2) \leq \chi'_s(4)$.*

Note that the suitable blown-up five-vertex cycles mentioned earlier certify that Theorem 3.1.21 is sharp and that $\chi'_s(4) \geq 20$. Theorem 3.1.2 extends, in 1, a result independently of Andersen [7] and Horák, Qing and Trotter [61], and, in 2, a result of Cranston [31]. These earlier results proved the unconditional bounds of Theorem 3.1.2 in the special case of G the line graph $L(F)$ of some (multi)graph F.

It is worth contrasting the work here and in [63] with the extremal study of $\chi(G)$ in terms of $\omega(G)$ where in general the situation for claw-free G is markedly different from and more complex than that for G the line graph of some (multi)graph, cf. [29].

In fact, for both $\omega(G) \in \{3, 4\}$ we show that Conjecture 3.1.1 reduces to the special case of G the line graph of a simple graph. The techniques we use for bounding $\chi(G^2)$ are purely combinatorial. They also apply when $\omega(G) > 4$ (as we describe just below), but seem to be most useful when $\omega(G)$ is small. It is natural that different methods are applicable in the small $\omega(G)$ versus large $\omega(G)$ cases, especially since this is also true of progress to date in the Erdős–Nešetřil conjecture.

Naturally, one could ask, for what (small) values of $\omega(G)$ does it remain true that Conjecture 3.1.1 is "equivalent" to the original conjecture of Erdős and Nešetřil? In light of the work in [63], it is conceivable that structural methods such as in [27, 29] will be helpful for this question. As an extremely modest step in this direction, we have shown the following reduction for $\omega(G) \geq 5$.

Theorem 3.1.3. *Fix $\omega \geq 5$. Then $\chi(G^2) \leq \max\{\chi'_s(\omega), 2\omega(\omega-1) - 3\}$ for every claw-free graph G with $\omega(G) = \omega$.*

To be transparent, let us compare this with one of the results from [63].

Theorem 3.1.4 (de Joannis de Verclos, Kang and Pastor [63])**.** *Fix $\omega \geq 5$. Then $\chi(G^2) \leq \max\{\chi'_{s,m}(\omega), 31\}$ for every claw-free graph G with $\omega(G) = \omega$.*

Combined with our Theorem 3.1.2 this implies that, in terms of reducing Conjecture 3.1.1 to those G which are multigraph line graphs, only the case $\omega(G) = 5$ remains with margin 2.

We remark that, for the conjecture of Erdős and Nešetřil itself when $\omega(G) \in \{5, 6, 7\}$ there has been little progress: respectively, a trivial bound based on the maximum degree of G^2 yields 41, 61, 85, Cranston [31] speculates that 37, 56, 79 are within reach, and the conjectured values are 29, 45, 58.

It gives insight to notice that the claw-free graphs with clique number at most ω are precisely those graphs each of whose neighbourhoods induces a subgraph with no clique of size $\omega - 1$ and no stable set of size 3. So a good understanding of the graphs that certify small off-diagonal Ramsey numbers can be useful for this class of problems.

Organisation: In the next section and Section 3.3, we introduce some basic tools we use. In Section 3.4, we treat the case $\omega(G) = 3$ and prove Theorem 3.1.21. In Section 3.5, we treat the case $\omega(G) = 4$ and prove Theorem 3.1.22. In Section 3.6, we briefly consider the extension of our methods to the case $\omega(G) \geq 5$ and prove Theorem 3.1.3.

3.2 Notation and preliminaries

We use standard graph theoretic notation. For instance, if v is a vertex of a graph G, then the neighbourhood of v is denoted by $N_G(v)$, and its degree by $\deg_G(v)$. For a subset S of vertices, we denote the neighbourhood of S by $N_G(S)$ and this is always assumed to be open, i.e. $N_G(S) = \cup_{s \in S} N_G(s) \setminus S$. We omit the subscripts if this causes no confusion. We frequently make use of the following simple lemmas.

Recall that the Ramsey number $R(k, \ell)$ is the minimum n such that in any graph on n vertices there is guaranteed to be a clique of k vertices or a stable set of ℓ vertices.

Lemma 3.2.1. *Let $G = (V, E)$ be a claw-free graph. For any $v \in V$, the induced subgraph $G[N(v)]$ contains no clique of $\omega(G)$ vertices and no stable set of 3 vertices. In particular, $\deg(v) < R(\omega(G), 3)$.*

Proof. If not, then with v there is either a clique of $\omega(G) + 1$ vertices or a claw. □

Lemma 3.2.2. *Let $G = (V, E)$ be a claw-free graph. For any $v, w \in V$ and $vw \in E$, any two distinct $x, y \in N(w) \setminus (\{v\} \cup N(v))$ are adjacent. In particular, $|N(w) \setminus (\{v\} \cup N(v))| \leq \omega(G) - 1$.*

Proof. If not, then v, w, x, y form a claw. So $\{w\} \cup N(w) \setminus (\{v\} \cup N(v))$ is a clique. □

It is not required next that $x, y \in N(v)$, but it is the typical context in which it is used.

Lemma 3.2.3. *Let $G = (V, E)$ be a claw-free graph. For any $v \in V$ and $w \in N(v)$, if $N(v) \cap N(w)$ contains two non-adjacent vertices x and y, then for any $z \in N(w) \setminus (\{v\} \cup N(v))$, either $xz \in E$ or $yz \in E$.*

Proof. If not, then w, x, y, z form a claw. □

3.3 A greedy procedure

In this section, we describe a general inductive procedure to use vertices of small square degree to colour squares in a class of graphs. This slightly refines a procedure in [63] so that it is suitable for our specific purposes.

Lemma 3.3.1. *Let K be a non-negative integer. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are graph classes such that $\mathcal{C}_1$ is closed under vertex deletion and every graph $G \in \mathcal{C}_2$ satisfies $\chi(G^2) \leq K+1$. Furthermore, suppose there exists $K' \leq K$ such that every graph $G \in \mathcal{C}_1$ satisfies one of the following:*

1. *G belongs to $\mathcal{C}_2$;*
2. *there is a vertex $v \in V(G)$ such that $\deg_{G^2}(v) \leq K'$, there is a vertex $x^* \in N_G(v)$ with $\deg_{G^2}(x^*) \leq K'+1$ and the set of all vertices $x \in N_G(v)$ with $\deg_{G^2}(x) > K'+2$ induces a clique in $(G \setminus v)^2$; or*
3. *there is a vertex $v \in V(G)$ such that $\deg_{G^2}(v) \leq K'$ and the set of all vertices $x \in N_G(v)$ with $\deg_{G^2}(x) > K'+1$ induces a clique in $(G \setminus v)^2$.*

For any $G \in \mathcal{C}_1$, $\chi(G^2) \leq K+1$.

Proof. We proceed by induction on the number of vertices. Since K is non-negative and the singleton graph is in $\mathcal{C}_1$, the base case of the induction holds. Let G be a graph in $\mathcal{C}_1$ with at least two vertices and suppose that the claim holds for any graph of $\mathcal{C}_1$ with fewer vertices than G has. If $G \in \mathcal{C}_2$, then we are done by the assumption on $\mathcal{C}_2$. So it only remains to consider the second and third possibility.

We now prove the bound under assumption of case 2. Let v be the vertex guaranteed in this case and write B for the set of vertices $x \in N_G(v)$ with $\deg_{G^2}(x) > K'+2$ and $S = N(v) \setminus B$. Since $\mathcal{C}_1$ is closed under vertex deletion, by induction there is a proper colouring φ of $(G \setminus v)^2$ with at most $K+1$ colours. Since B is a clique, all elements in B are assigned different colours under φ. From φ, we will now obtain a new proper $(K+1)-$colouring φ' of $(G \backslash v)^2$ such that *all* elements of $N_G(v)$ have different colours.

First we uncolour all vertices in S. We then wish to recolour them with pairwise distinct colours as follows. Given $s \in S$, we say a colour in $\{1, \ldots, K+1\}$ is *available* to s if it is distinct from any colour assigned by φ to the vertices in $N_{G^2}(s) \backslash (\{v\} \cup S)$. Since $\deg_{G^2}(s) \leq K'+2 \leq K+2$ and $\{v\} \cup S \setminus \{s\} \subseteq N_{G^2}(s)$, the number of colours available to s is at least $K+1-(\deg_{G^2}(s) - |\{v\} \cup S \setminus \{s\}|) \leq K+1-((K+2)-|S|) = |S|-1$. Furthermore, since $x^* \in S$ and $\deg_{G^2}(x^*) \leq K'+1 \leq K+1$, the number of colours available to x^* is at least $|S|$. Since the complete graph on $|S|$ vertices is (greedily) list colourable for any list assignment with $|S|-1$ lists of size $|S|-1$ and one list of size $|S|$, it follows that we can recolour the vertices of S with pairwise distinct available colours.

This new colouring φ' is a proper $(K+1)$-colouring of $(G \backslash v)^2$ such that all elements in $N_G(v)$ have different colours. Since $\deg_{G^2}(v) \leq K' \leq K$, there is at least one colour not appearing in $N_{G^2}(v)$ that we can assign to v so that together with φ' we obtain a proper $(K+1)$-colouring of G^2.

The proof under assumption 3 is nearly the same as under 2. Defining B for the set of vertices $x \in N_G(v)$ with $\deg_{G^2}(x) > K'+1$ and $S = N(v) \backslash B$, we obtain that every $s \in S$ has $|S|$ available colours. This allows us to complete the colouring as before. $\square$

3.4 Clique number three

In this section, we prove Theorem 3.1.21. We actually prove the following result.

Theorem 3.4.1. *Let $G = (V, E)$ be a connected claw-free graph with $\omega(G) = 3$. Then one of the following is true:*

1. *G is the icosahedron;*
2. *G is the line graph $L(F)$ of a 3-regular graph F; or*
3. *there exists $v \in V$ with $\deg_{G^2}(v) \leq 9$ such that $\deg_{G^2}(x) \leq 11$ for all $x \in N_G(v)$. Furthermore, either there exists $x^* \in N_G(v)$ with $\deg_{G^2}(x^*) \leq 10$, or $N_G(v)$ induces a clique in $(G\backslash v)^2$.*

Let us first see how this easily implies Theorem 3.1.21.

Proof of Theorem 3.1.21. Let $\mathcal{C}_1$ be the class of claw-free graphs G with $\omega(G) \leq 3$. Clearly $\mathcal{C}_1$ contains the singleton graph and is closed under vertex deletion.

Let $\mathcal{C}_2$ be the class of graphs formed by taking all claw-free graphs G with $\omega(G) \leq 2$, the icosahedron, and the line graphs $L(F)$ of all 3-regular graphs F. If G is a claw-free graph with $\omega(G) \leq 2$, then $\chi(G^2) \leq 5$. If G is the icosahedron, then $\chi(G^2) \leq 6$ is certified by giving every pair of antipodal points the same colour. If G is the line graph of a 3-regular graph, then $\chi(G^2) \leq 10$ by the strong edge-colouring result due, independently, to Andersen [7] and to Horák, Qing and Trotter [61].

Theorem 3.4.1 certifies that we can apply Lemma 3.3.1 with $K = K' = 9$. □

Proof of Theorem 3.4.1. First we show that either case 1 or 2 applies, or that there exists a vertex $v \in V$ with $\deg_{G^2}(v) \leq 9$. At the end, we show that, for all such v, it also holds that $\deg_{G^2}(x) \leq 11$ for all $x \in N_G(v)$ and that furthermore these vertices either induce a clique in $(G\backslash v)^2$, or contain a vertex x^* with $\deg_{G^2}(x^*)$ at most 10.

First note that the maximum degree $\Delta(G)$ of G is at most 5. This follows from Lemma 3.2.1 and the fact that $R(3, 3) = 6$. Moreover, note that, for any $v \in V$ with $\deg(v) = 5$, $G[N(v)]$ must be a 5-cycle by Lemma 3.2.1.

For $v \in V$ with $\deg(v) \leq 2$, we have $\deg_{G^2}(v) \leq 2 + 2 \cdot 2 = 6$ by Lemma 3.2.2. For $v \in V$ with $\deg(v) = 3$, we have $\deg_{G^2}(v) \leq 3 + 3 \cdot 2 = 9$ by Lemma 3.2.2. So in terms of proving the existence of a vertex v with $\deg_{G^2}(v) \leq 9$, we can assume hereafter that the minimum degree of G satisfies $\delta(G) \geq 4$.

For $v \in V$ with $\deg(v) = 4$, we call v *good* if the subgraph $G[N(v)]$ induced by $N(v)$ is not the disjoint union of two edges. Assume for the moment that G contains no good vertex.

If $\delta(G) = \Delta(G) = 4$, then every neighbourhood induces the disjoint union of two cliques (each of exactly two vertices). Recall that a graph is the line graph of a graph if its edges can be partitioned into maximal cliques so that no vertex belongs to more than two such cliques and additionally, no two vertices are both in the same two cliques. We can designate the maximal cliques as follows: for $v \in V$ and a clique C that is maximal in $N(v)$, designate $v \cup C$ as a maximal clique for the requisite edge partition. Indeed, every edge v_1v_2 is designated as part of one of the cliques, either from the perspective of v_1 or of v_2. Moreover, the clique to which v_1v_2 is designated does not differ depending on the endpoint from which the perspective is taken, since every neighbourhood induces the disjoint union of two cliques. As each of the designated cliques has exactly three vertices, it follows that G is the line graph $L(F)$ of a 3-regular graph F.

If, on the other hand, there exists $v \in V$ with $\deg(v) = 5$, then consider $x \in N(v)$. Since $G[N(v)]$ is a 5-cycle, x has three neighbours y_1, v, y_3 that induce a 3-vertex path y_1vy_3. This means $G[N(x)]$ is not the union of two cliques. By our assumption that no vertex is good, it follows that x has degree 5. So G is the icosahedron, the unique connected graph in which every neighbourhood induces a 5-cycle. (Uniqueness can be easily seen by constructing the graph up to distance 2 from v in the only possible way respecting induced 5-cycles, and then noting that the vertices at distance 2 from v induce a 5-cycle and that they all need to be adjacent to a 12th and final vertex.)

From now on, let $v \in V$ be a good vertex. We next show that $|N(N(v)) \setminus \{v\}| \leq 5$ (which implies $\deg_{G^2}(v) \leq 9$).

Since $G[N(v)]$ has no stable set of three vertices and v is good, $G[N(v)]$ has at least three edges. Moreover, since $G[N(v)]$ has no clique of three vertices, we can write $N(v) = \{x_1, x_2, x_3, x_4\}$ such that $x_1x_2, x_2x_3, x_3x_4 \in E$ and $x_1x_3, x_2x_4 \notin E$. By Lemma 3.2.2, both x_1 and x_4 have at most 2 neighbours outside $\{v\} \cup N(v)$. So it suffices to show that $\{x_2, x_3\}$ cannot have two neighbours outside $\{v\} \cup N(v)$ which are not neighbours of $\{x_1, x_4\}$. By contradiction, let p, q be these vertices. Without loss of generality, p is a neighbour of x_2. Then p is ajdacent to x_3, for otherwise $x_1x_2x_3p$ would be a claw. Similarly, q is adjacent to both x_2 and x_3. But then pq is an edge (otherwise x_1pqx_2 would be a claw), so that x_2x_3pq is a K_4. Contradiction. This concludes the proof that there exists a vertex v with $\deg_{G^2}(v) \leq 9$.

From now on, let v be one of the vertices for which we showed above that $\deg_{G^2}(v) \leq 9$. In particular, if v has degree 4 then it is a good vertex.

Let us call a vertex x *extremely bad* if $\deg_{G^2}(x) \geq 12$. We already observed that no vertex x with $\deg(x) \leq 3$ is extremely bad. If $\deg(x) = 5$, then $N(x)$ induces a 5-cycle and so by Lemma 3.2.3 every vertex in $N(N(x)) \setminus \{x\}$ has at least two neighbours in $N(x)$, so $|N(N(x)) \setminus \{x\}| \leq 5$. So a vertex x can only be very bad if $\deg(x) = 4$ and it is not good. In particular, by Lemma 3.2.2, not only does the neighbourhood of x induce a disjoint union of two edges, but also the same is true for every neighbour of x. This implies that $N(v)$ does not contain an extremely bad vertex.

Let us call a vertex x *very bad* if $\deg_{G^2}(x) = 11$. We are done if there exists $x^* \in N_G(v)$ with $\deg_{G^2}(x^*) \leq 10$. So we may assume from now on that *all* vertices in $N_G(v)$ are very bad, and we need to show that they induce a clique in $(G \setminus v)^2$. Assume for a contradiction that they do not. Since the neighbourhood of a degree 5 vertex induces a 5-cycle, of which the square is a clique, we may assume that $\deg(v) \leq 4$. If $\deg(v) = 3$, then there are $x_1, x_2, x_3 \in N(v)$ such that $x_1x_2, x_2x_3 \notin E(G)$, so $\deg_G(x_2) \leq 3$, so $\deg_{G^2}(x_2) \leq 9$, contradicting that x_2 is very bad. Similarly if $\deg(v) \leq 2$. Thus we have reduced to the case that v is a good vertex (of degree 4). As argued before, we can then write $N(v) = \{x_1, x_2, x_3, x_4\}$ such that $x_1x_2, x_2x_3, x_3x_4 \in E$ and $x_1x_3, x_2x_4 \notin E$. Since $N(v)$ does not induce a clique in $(G \setminus v)^2$, it follows that also $x_1x_4 \notin (E)$. Therefore $\deg^2_G(x_1) \leq 10$, contradicting that x_1 is very bad. This completes the proof. □

3.5 Clique number four

The proof of Theorem 3.4.1 suggests the following rougher but more general phenomenon. This follows from Lemmas 3.2.2 and 3.2.3 together with a double-counting argument.

For $G = (V, E)$ and $v \in V$, we define the following subset of $N(v)$:

$$Z(v) := \{w \in N(v) \mid \exists x, y \in N(v) \text{ such that } xw, wy \in E \text{ and } xy \notin E\}.$$

Lemma 3.5.1. *Let $G = (V, E)$ be a claw-free graph. For any $v \in V$,*

$$\begin{aligned}|N(N(v)) \setminus \{v\}| &\leq \sum_{w \in N(v)\setminus Z(v)} |N(w) \setminus (\{v\} \cup N(v))| + \frac{1}{2} \sum_{w \in Z(v)} |N(w) \setminus (\{v\} \cup N(v))| \\ &\leq \left(\deg(v) - \frac{1}{2}|Z(v)|\right)(\omega(G) - 1).\end{aligned}$$

Proof. Let $w \in Z(v)$. By Lemma 3.2.3, any $x \in N(w) \setminus (\{v\} \cup N(v))$ also satisfies $x \in N(y) \setminus (\{v\} \cup N(v))$ for some $y \in N(v) \setminus \{w\}$. So

$$|N(N(v)) \setminus \{v\}| = \sum_{w \in N(v)} \sum_{x \in N(w)\setminus(\{v\}\cup N(v))} \frac{1}{|\{u \in N(v) \mid x \in N(u)\}|}$$

is at most $\sum_{w \in N(v)\setminus Z(v)} |N(w) \setminus (\{v\} \cup N(v))| + \frac{1}{2} \sum_{w \in Z(v)} |N(w) \setminus (\{v\} \cup N(v))|$. Now apply Lemma 3.2.2. □

This has the following immediate consequence.

Corollary 3.5.2. *Let $G = (V, E)$ be a claw-free graph. For any $v \in V$ with $\deg(v) \geq 2\omega(G) - 1$, we have $Z(v) = N(v)$ and therefore*

$$|N(N(v)) \setminus \{v\}| \leq \frac{1}{2} \sum_{w \in N(v)} |N(w) \setminus (\{v\} \cup N(v))| \leq \frac{1}{2} \deg(v)(\omega(G) - 1).$$

Proof. Let $w \in N(v)$ and consider $N_{G[N(v)]}(w)$. By Lemma 3.2.2, $\deg_{G[N(v)]}(w) \geq \deg(v) - (\omega(G) - 1) - 1 \geq \omega(G) - 1$. Then $N_{G[N(v)]}(w)$ contains a pair of non-adjacent vertices, or else $\{v, w\} \cup N_{G[N(v)]}(w)$ is a clique of $\omega(G)+1$ vertices. As w was arbitrary, we have just shown that $Z(v) = N(v)$. So the result follows from Lemma 3.5.1. □

We now prove the following result. Similarly to what we saw if $\omega(G) = 3$, this implies for any claw-free G with $\omega(G) = 4$ that $\chi(G^2) \leq 22$ by Lemma 3.3.1 with $K = 21$ and $K' = 19$, due to a result of Cranston [31]. Furthermore, since $\chi'_s(4) \geq 20$, we may make the choice $K = \chi'_s(4) - 1$ and $K' = 19$ to obtain Theorem 3.1.22, i.e. that Conjecture 3.1.1 for $\omega(G) = 4$ reduces to the corresponding case of the Erdős–Nešetřil conjecture.

Theorem 3.5.3. *Let $G = (V, E)$ be a connected claw-free graph with $\omega(G) = 4$. Then one of the following is true:*

1. *G is the line graph $L(F)$ of a graph F of maximum degree 4; or*

2. *there exists $v \in V$ with $\deg_{G^2}(v) \leq 19$ such that the set of all vertices $x \in N_G(v)$ with $\deg_{G^2}(x) \geq 21$ induces a clique in $(G \setminus v)^2$.*

Proof. First we show that either case 1 applies or that there exists a vertex $v \in V$ with $\deg_{G^2}(v) \leq 19$. At the end, we show that, for all such v, it also holds that the set of vertices $x \in N_G(v)$ with $\deg_{G^2}(x) \geq 21$ induces a clique in $(G \setminus v)^2$.

First note that the maximum degree $\Delta(G)$ of G is at most 8. This follows from Lemma 3.2.1 and the fact that $R(4,3) = 9$.

For $v \in V$ with $\deg(v) \leq 4$, we have $\deg_{G^2}(v) \leq 4 + 4 \cdot 3 = 16$ by Lemma 3.2.2.

Note that, for $v \in V$ with $\deg(v) = 5$, we have $\deg_{G^2}(v) \leq 5 + 5 \cdot 3 = 20$ by Lemma 3.5.1, but equality cannot occur here unless $Z(v) = \emptyset$. (Indeed, if $Z(v) \neq \emptyset$, then

For $v \in V$ with $\deg(v) = 5$ and $Z(v) = \emptyset$, $G[N(v)]$ is the disjoint union of cliques, and in particular it must be the disjoint union of an edge and a triangle.

For $v \in V$ with $\deg(v) = 7$, we have $\deg_{G^2}(v) \leq 7 + 21/2 = 17.5$ by Corollary 3.5.2.

Let $v \in V$ with $\deg(v) = 8$. By Corollary 3.5.2, $Z(v) = N(v)$ and so we already have $\deg_{G^2}(v) \leq 8 + 24/2 = 20$, but we want one better. Let $w \in N(v)$. By Lemma 3.2.2, $N(v) \setminus (N_{G[N(v)]}(w) \cup \{w\})$ is a clique, so $\deg_{G[N(v)]}(w) \geq \deg(v) - \omega(G) = 4$. Now $N_{G[N(v)]}(w)$ contains no clique or stable set of three vertices, or else G contains a clique of 5 vertices or a claw. We can therefore find four vertices $x_1, x_2, x_3, x_4 \in N_{G[N(v)]}(w)$ such that $x_1x_2, x_3x_4 \notin E$. (There is at least one non-edge among x_1, x_2, x_3, say, x_1x_2. Since G is claw-free at least one of x_1x_3 and x_2x_3 is an edge, say, x_2x_3. Among x_2, x_3, x_4, there is at least one non-edge, which together with x_1x_2 or x_1x_3 forms a two-edge matching in the complement, which is what we wanted, after relabelling.) By Lemma 3.2.3, for every $y \in N(w) \setminus (\{v\} \cup N(v))$, either $x_1y \in E$ or $x_2y \in E$ *and* $x_3y \in E$ or $x_4y \in E$. We have just shown that every vertex in $N(N(v)) \setminus \{v\}$ has at least three neighbours in $N(v)$. Therefore, $|N(N(v)) \setminus \{v\}| \leq \frac{1}{3}\deg(v)(\omega(G) - 1) = 8$ and $\deg_{G^2}(v) \leq 16$.

Let $v \in V$ with $\deg(v) = 6$. By Lemma 3.2.2, the minimum degree of $G[N(v)]$ satisfies $\delta(G[N(v)]) \geq \deg(v) - \omega(G) = 2$. Since G contains no clique of 5 vertices, every vertex with degree at least 3 in $G[N(v)]$ must also be in $Z(v)$. So we know there are at most two such vertices, or else by Lemma 3.5.1 $\deg_{G^2}(v) \leq 6 + \lfloor (6-3/2) \cdot 3 \rfloor = 19$. First suppose there is a vertex w with degree 5 in $G[N(v)]$. Since $N_{G[N(v)]}(w)$ contains no clique or stable set of three vertices, it must be that $G[N(v)]$ consists of w adjacent to all vertices of a 5-cycle, in which case all six vertices have degree at least 3 in $G[N(v)]$. This contradicts that at most two vertices of degree at least 3 are allowed in $G[N(v)]$. Next suppose that there is a vertex w with degree 4 in $G[N(v)]$. Then there exists $w' \in N(v)$ with $ww' \notin E$. As we argued in the last paragraph, there exist $x_1, x_2, x_3, x_4 \in N_{G[N(v)]}(w)$ such that $x_1x_2, x_3x_4 \notin E$. Since G is claw-free, it must be that w' is adjacent to one of x_1 and x_2 and also to one of x_3 and x_4; without loss of generality suppose $x_1w', x_3w' \in E$. It follows that x_1, x_3, w are three vertices with degree at least 3 in $G[N(v)]$, which was not allowed. So now we have reduced to the case where $2 \leq \delta(G[N(v)]) \leq \Delta(G[N(v)]) \leq 3$ and there are at most two vertices with degree 3 in $G[N(v)]$. Since G is claw-free, there are only two possibilities for the structure of $G[N(v)]$: either it is a disjoint union of two triangles, or it is that graph with the inclusion of exactly one additional edge.

We call a vertex v *good* if its neighbourhood structure does not satisfy one of the following:

- $G[N(v)]$ is the disjoint union of a singleton and a triangle;

- $G[N(v)]$ is the disjoint union of an edge and a triangle;
- $G[N(v)]$ is the disjoint union of an edge and a triangle plus one more edge;
- $G[N(v)]$ is the disjoint union of two triangles;
- $G[N(v)]$ is the disjoint union of two triangles plus one more edge; or
- $G[N(v)]$ is the disjoint union of two triangles plus two more non-incident edges.

Recall that a graph is the line graph of a graph if its edges can be partitioned into maximal cliques so that no vertex belongs to more than two such cliques and additionally, no two vertices are both in the same two cliques. If no vertex $v \in V$ is good, then we can designate the maximal cliques as follows: for each $v \in V$ and for any C one of the two maximum cliques of $G[N(v)]$ specified in one of the cases above (this is well-defined), we designate $v \cup C$ as a maximal clique for the requisite edge partition. Indeed, every edge v_1v_2 is designated as part of one of the cliques, either from the perspective of v_1 or of v_2. Moreover, the clique to which v_1v_2 is designated does not differ depending on the endpoint from which the perspective is taken, by a brief consideration of the six impermissible neighbourhood structures defining a good vertex. As each of the designated cliques has at most four vertices, it follows that in this case G is the line graph $L(F)$ of a graph F of maximum degree 4.

Our case analysis has shown that either no vertex of G is good, in which case G is the line graph of a graph of maximum degree 4, or there is some good $v \in V$ with $\deg_{G^2}(v) \leq 19$. From now on, we fix one such good vertex v.

Let us call a vertex x *very bad* if $\deg_{G^2}(x) \geq 21$. We already observed that x must then have $\deg(x) = 6$. By the case analysis above, the neighbourhood of x either induces a disjoint union of two triangles or is that graph plus one more edge. However, the latter case is excluded, as we will now demonstrate. Suppose the neighbourhood of a vertex x induces two triangles $w_1w_2w_3$ and $w_4w_5w_6$ plus one more edge w_1w_4. Our goal is to derive then that $\deg_{G^2}(x) \leq 20$, so that x cannot be very bad. By Lemma 3.2.2, w_i has at most three neighbours outside $\{v\} \cup N(v)$, for all $i \in \{2, 3, 5, 6\}$. So it suffices to show that $\{w_1, w_4\}$ cannot have three neighbours outside $\{v\} \cup N(v)$ which are not a neighbour of $\{w_2, w_3, w_5, w_6\}$. By contradiction, let p, q, r be these neighbours. Without loss of generality, p is a neighbour of w_1. Then p is also adjacent to w_4 (otherwise claw). The same argument applies to q and r, so that $\{p, q, r\}$ must be complete to $\{w_1, w_4\}$. Furthermore, by claw-freeness, pqr must be a triangle. But then $\{w_1, w_4, p, q, r\}$ induces a K_5. Contradiction. This completes the proof that the neighbourhood of a very bad vertex induces the disjoint union of two triangles.

Let x_1 be a very bad vertex in $N(v)$. Since $N(x_1)$ induces two disjoint triangles (one containing v) it follows that x_1 is part of a triangle $x_1x_2x_3$ in $N(v)$ and there is no edge between x_1 and $N(v) \setminus \{x_1, x_2, x_3\}$. Thus each vertex in $N(v) \setminus \{x_1, x_2, x_3\}$ is at distance exactly 2 from x_1 (with respect to G) so that $N(v) \setminus \{x_1, x_2, x_3\}$ is a clique by Lemma 3.2.2.

Suppose now that the very bad vertices in $N(v)$ do not form a clique in $(G \setminus v)^2$. Writing $N(v) := \{x_1, \ldots, x_6\}$, then there exist two very bad vertices x_1, x_6, say, that are at distance greater than 2 in $G \setminus v$. By the previous paragraph, $N(v)$ is covered by two disjoint triangles. Because v is good, it follows (up to symmetry of x_1 and x_6)

that the following is a subgraph of the graph induced by $N(v)$: two disjoint triangles $x_1x_2x_3$ and $x_4x_5x_6$ plus two edges x_2x_4, x_3x_4. Note that x_2, x_3 and v are neighbours of x_1 that have a common neighbour at distance 2 from x_1, namely x_4, and separately from that, x_2 and x_3 have a common neighbour in $N(N(x_1)) \cap N(N(v)) \setminus \{v, x_1\}$. It follows that $\deg_{G^2}(x_1) \leq 20$, contradicting that x_1 is very bad. We have shown that the very bad vertices in $N(v)$ form a clique in $(G \setminus v)^2$ and this concludes the proof. □

3.6 Clique number at least five

The proof of Theorem 3.5.3 suggests the following refinement of Lemma 3.5.1. This could be useful towards reductions to the line graph setting for $\omega(G) \geq 5$.

For $G = (V, E)$ and $v \in V$ and $w \in N(v)$, we define $q(w)$ to be the matching number of the complement of $G[N_{G[N(v)]}(w)]$. Note that $q(w) \geq 1$ if and only if $w \in Z(v)$.

Lemma 3.6.1. *Let $G = (V, E)$ be a claw-free graph. For any $v \in V$,*

$$|N(N(v)) \setminus \{v\}| \leq \sum_{w \in N(v)} \frac{|N(w) \setminus (\{v\} \cup N(v))|}{q(w) + 1} \leq (\omega(G) - 1) \sum_{w \in N(v)} \frac{1}{q(w) + 1}.$$

Proof. Let $a_1b_1, a_2b_2, \ldots, a_{q(w)}b_{q(w)}$ be edges of a maximum matching in the complement of $G[N_{G[N(v)]}(w)]$. Note that w and $a_1, b_1, \ldots, a_{q(w)}, b_{q(w)}$ are all distinct vertices in $N(v) \cap N(w)$. Let $x \in N(w) \setminus \{v\}$. For all $i \in \{1, \ldots, q(w)\}$, it holds that $wa_i, wb_i \in E$ and $a_ib_i \notin E$, so by Lemma 3.2.3 x is not only a neighbour of w, but also a neighbour of a_i or b_i. This implies that $|\{u \in N(v) \mid x \in N(u)\}| \geq q(w) + 1$. So

$$|N(N(v)) \setminus \{v\}| = \sum_{w \in N(v)} \sum_{x \in N(w) \setminus (\{v\} \cup N(v))} \frac{1}{|\{u \in N(v) \mid x \in N(u)\}|}$$

is at most $\sum_{w \in N(v)} |N(w) \setminus (\{v\} \cup N(v))|/(q(w) + 1)$. Now apply Lemma 3.2.2. □

Lemma 3.6.1 yields the following corollary.

Corollary 3.6.2. *Let $G = (V, E)$ be a claw-free graph with $\omega(G) \geq 4$. For any $v \in V$ with $\deg(v) \geq 2\omega(G) - 1$,*

$$|N(N(v)) \setminus \{v\}| \leq \frac{\deg(v)(\omega(G) - 1)}{\lceil (\deg(v) + 1)/2 \rceil + 2 - \omega(G)}.$$

Proof. Let $w \in N(v)$. It suffices to establish a suitable lower bound for $q(w)$. By Lemma 3.2.2, $\deg_{G[N(v)]}(w) \geq \deg(v) - \omega(G) \geq \omega(G) - 1$, and so in any subset of $N_{G[N(v)]}(w)$ with at least $\omega(G) - 1$ vertices there must be at least one non-edge (or else G has a clique of $\omega(G) + 1$ vertices). So we can iteratively extract two vertices from $N_{G[N(v)]}(w)$ that form an edge of the complement of $G[N_{G[N(v)]}(w)]$ until at most $\omega(G) - 2$ vertices remain. It follows that

$$\begin{aligned} q(w) &\geq \left\lceil \frac{1}{2}(\deg_{G[N(v)]}(w) - (\omega(G) - 2)) \right\rceil \geq \left\lceil \frac{1}{2}(\deg(v) - \omega(G) - (\omega(G) - 2)) \right\rceil \\ &= \lceil \deg(v)/2 \rceil + 1 - \omega(G). \end{aligned}$$

If $\deg(v)$ is even, then after we have extracted $\lceil \deg(v)/2 \rceil - \omega(G)$ pairs as above at least $\omega(G)$ vertices remain, call them $x_1, \dots, x_{\omega(G)}$. Among $x_1, \dots, x_{\omega(G)-1}$ there is at least one non-edge, say, $x_1x_2 \notin E$ without loss of generality.

Since $\omega(G) \geq 4$, there is at least one non-edge ab among $x_2, \dots, x_{\omega(G)}$, and at least one non-edge cd among $x_1, x_3, \dots x_{\omega(G)}$. The non-edges x_1x_2, ab and cd may not form a stable set of size three, since otherwise there would be a claw. Therefore at least two of them comprise a two-edge matching in the complement of $G[\{x_1, \dots, x_{\omega(G)}\}]$. So indeed we have for any parity of $\deg(v)$ that

$$q(w) \geq \lceil (\deg(v)+1)/2 \rceil + 1 - \omega(G).$$

As w was arbitrary, the result now follows from Lemma 3.6.1. □

Let us now make explicit some general consequence of Corollary 3.6.2. An awkward but routine optimisation checks that for $k \geq 5$ and $x \in \{2k-1, 2k, \dots\}$, the expression $f(x) := x + \frac{x(k-1)}{\lceil (x+1)/2 \rceil + 2 - k}$ is maximised with $x = 2k-1$ or with $x \in \{y, y+1\}$ for y as large as possible. (This follows e.g. from the facts that $f(2k-1) > f(2k)$ and that there is some $x_0 > 2k-1$ such that the derivative of $f^*(x) := x + \frac{x(k-1)}{(x+1)/2+2-k}$ is negative for all $2k-1 \leq x < x_0$ and positive for all $x > x_0$.) By Lemma 3.2.1, $R(\omega(G),3)-1$ and $R(\omega(G),3)-2$ are the two largest allowed values of $\deg(v)$. So by Corollary 3.6.2, if v is a vertex of a claw-free graph G with $\deg(v) \geq 2\omega(G)-1$, then $\deg_{G^2}(v) \leq \max\{f(2\omega(G)-1), f(R(\omega(G),3)-1), f(R(\omega(G),3)-2)\}$, yielding

$$\begin{aligned}\deg_{G^2}(v) \leq \max\Bigg\{ & 2\omega(G) - 1 + (\omega(G)-1/2)(\omega(G)-1), \\ & R(\omega(G),3) - 2 + \frac{(R(\omega(G),3)-2)(\omega(G)-1)}{(R(\omega(G),3)-1)/2 + 2 - \omega(G)}, \\ & R(\omega(G),3) - 1 + \frac{(R(\omega(G),3)-1)(\omega(G)-1)}{R(\omega(G),3)/2 + 2 - \omega(G)} \Bigg\}.\end{aligned} \tag{3.1}$$

Moreover, (3.1) remains valid when we substitute $R(\omega(G),3)$ with any upper bound. It is known [40] that $R(\omega(G),3) \leq \binom{\omega(G)+1}{2}$. With this and some routine calculus, (3.1) implies that $\deg_{G^2}(v) \leq 2\omega(G)(\omega(G)-1)$ provided $\omega(G) \geq 3$. Since those v with $\deg(v) \leq 2\omega(G)-2$ have $\deg_{G^2}(v) \leq 2\omega(G)(\omega(G)-1)$ by Lemma 3.2.2, we have the following "trivial" bound on $\chi(G^2)$. This was proved not via $\Delta(G^2)$ but by a different method in [63].

Corollary 3.6.3. *If G is a claw-free graph, then $\chi(G^2) \leq \Delta(G^2)+1 \leq 2\omega(G)(\omega(G)-1)+1$.*

Also (3.1) implies that, if v is a vertex of a claw-free graph G with $\deg(v) \geq 2\omega(G)-1$, then $\deg_{G^2}(v) \leq \frac{1}{4}(5\omega(G)^2 - 2\omega(G)+1) - 1$ provided $\omega(G) \geq 5$. We use this for the following.

Theorem 3.6.4. *Let $G = (V,E)$ be a connected claw-free graph with $\omega(G) = \omega \geq 5$. Then one of the following is true:*

1. *G is the line graph $L(F)$ of a graph F of maximum degree ω; or*

2. *there exists* $v \in V$ *with* $\deg_{G^2}(v) \le 2\omega(\omega-1)-4$ *such that* $\deg_{G^2}(x) \le 2\omega(\omega-1)-3$ *for all* $x \in N_G(v)$.

Proof. By the last remark (which followed from Corollary 3.6.2), for $v \in V$ with $\deg(v) \ge 2\omega - 1$, we have that $\deg_{G^2}(v) \le \frac{1}{4}(5\omega^2 - 2\omega + 1) - 1 \le 2\omega(\omega - 1) - 4$ since $\omega \ge 5$.

For $v \in V$ with $\deg(v) \le 2\omega - 3$, we have by Lemma 3.2.2 that $\deg_{G^2}(v) \le \omega(2\omega - 3) \le 2\omega(\omega - 1) - 4$ since $\omega \ge 5$.

Let $v \in V$ with $\deg(v) = 2\omega - 2$. If $G[N(v)]$ is not the disjoint union of two cliques, then $|Z(v)| \ge 2$. (Clearly $|Z(v)| > 0$ if $G[N(v)]$ is not the disjoint union of two cliques, but if on the contrary $|Z(v)| = 1$ then let $w \in N(v)$ be the unique vertex such that there exist $x, y \in N(v)$ for which $xw, wy \in E$, $xy \notin E$. By the uniqueness of w, x does not have any neighbours in $N(v)$ in common with y. Moreover, $(\{x\} \cup N(x)) \cap N(v)$ is a clique, because otherwise we would either have a claw or $x \in Z(v)$. By the uniqueness of w, $(\{x\} \cup N(x)) \cap N(v) \subseteq N(w) \cup \{w\}$. The same arguments hold with the roles of x and y exchanged. It follows that $G[N(v)]$ is the union of two cliques with exactly one vertex in common. Since each clique in $G[N(v)]$ is of size at most $\omega - 1$, this is a contradiction to $\deg(v) = 2\omega - 2$.) It then follows by Lemma 3.5.1 that $\deg_{G^2}(v) \le (2\omega - 1)(\omega - 1) \le 2\omega(\omega - 1) - 4$ since $\omega \ge 5$.

We have shown that one of the following two possibilities must hold for G:

1. for every $v \in V$ it holds that $G[N(v)]$ is the disjoint union of two cliques of size $\omega - 1$ or that same graph with one extra edge between the two cliques, or the disjoint union of two cliques one of size $\omega - 2$ the other of size $\omega - 1$; or

2. there is some $v \in V$ with $\deg_{G^2}(v) \le 2\omega(\omega - 1) - 4$.

In the former situation, G is the line graph of a graph of maximum degree ω.

Let us call a vertex v *very bad* if $\deg_{G^2}(v) \ge 2\omega(\omega-1)-2$. We already observed that v must then have $\deg(v) = 2\omega-2$. As argued just above, Lemma 3.5.1 implies that the neighbourhood of v induces a disjoint union of two cliques of size $\omega-1$. Moreover, using Lemma 3.2.2, we have that for every neighbour x of v the neighbourhood of x induces the disjoint union of two cliques of size $\omega - 1$, or that same graph plus one more edge, or the disjoint union of two cliques one of size $\omega-2$ the other of size $\omega-1$. This implies that, for every vertex v for which we showed above that $\deg_{G^2}(v) \le 2\omega(\omega - 1) - 4$ (not including those cases corresponding to the promised line graph of maximum degree ω), it also holds that $N(v)$ does not contain a very bad vertex. This completes the proof. □

Proof of Theorem 3.1.3. Together with the trivial bound, Theorem 3.6.4 certifies that we can apply Lemma 3.3.1 with $K = K' = \max\{\chi'_s(\omega), 2\omega(\omega - 1) - 3\} - 1$. □

We wanted to illustrate how our methods could extend to larger values of $\omega(G)$. It is likely that Theorem 3.6.4 can be improved, particularly since we did not use the full strength of Lemma 3.3.1. On the other hand, since the Erdős–Nešetřil conjecture itself is open apart from the case of graphs of maximum degree at most 3, we leave this to further investigation.

Chapter 4

Strong cliques in cycle-free graphs

In support of a notorious conjecture of Erdős and Nešetřil, a classic result due to Faudree, Gýarfás, Schelp and Tuza is that in every bipartite graph of maximum degree Δ, there are at most Δ^2 edges in any set of edges every pair of which is either incident or connected by an edge, i.e. the graph has strong clique number at most Δ^2. We put forward four strengthened versions of this result. First, we show that every C_5-free multigraph has strong clique number at most Δ^2, and we also derive a strengthened bound in terms of the Ore-degree. Second, we show that every triangle-free graph of maximum degree Δ has strong clique number at most $\frac{5}{4}\Delta^2$, which is sharp due to blown-up C_5's. Third, we conjecture that in any graph of maximum degree $\Delta \geq 4$ that additionally contains no cycle of length $2k$, the strong clique number is at most $(2k-1)(\Delta-k+1)$, which would be sharp. We prove it for $k=2$ and for general k we derive a slightly worse bound. Fourth, we conjecture that in any bipartite graph of maximum degree $\Delta \geq 2$ that additionally contains no cycle of length $2k$, the strong clique number is at most $k\Delta+1-k$, which would be sharp if true. We provide some evidence towards this conjecture. Along the way, we also obtain that the strong clique number is essentially bounded by the product of $\Delta(G)$ and the Hadwiger number $h(G)$ of G.

4.1 Introduction

4.1.1 Strong chromatic index and strong clique number

The *strong chromatic index* $\chi(L(G)^2)$ of a graph G is the chromatic number of the square of the line graph of G. Equivalently, it is the minimum number of colours one needs in order to colour the edges of G, such that every two edges that are at distance ≤ 1 receive different colours.

Since $\Delta(L(G)^2) < 2\Delta(G)^2$, a trivial upper bound for the strong chromatic index is $\omega(L(G)^2) \leq 2\Delta(G)^2$. Erdős and Nešetřil [39] conjectured the following stronger bound.

Conjecture 4.1.1 ([39, 43])**.** *For any graph G of maximum degree Δ,*

$$\chi(L(G)^2) \leq \frac{5}{4}\Delta^2.$$

If true, Conjecture 4.1.1 is sharp for even Δ. This is exemplified by a *blown-up 5-cycle*, which is the graph obtained after replacing each vertex in a C_5 with a stable set of size k and replacing each edge ab of C_5 with a complete bipartite graph between the stable sets corresponding to a and b. This graph has maximum degree $2k$ and has $5/2k^2$ edges, which are all within distance 2 from each other. Therefore the square of its line graph is a clique and $\chi(L(G)^2) = 5/2k^2 = 5/4\Delta^2$. For odd Δ, Erdős and Nešetřil conjectured that $\chi(L(G)^2) \leq \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + 1$, which again would be sharp.

Molloy and Reed [86] combined a structural estimate with a probabilistic colouring method to find a fixed but very small $\epsilon > 0$ such that $\chi(L(G)^2) \leq (2-\epsilon)\Delta^2$ for all graphs. Bruhn and Joos [65] optimized this technique considerably and obtained $\chi(L(G)^2) \leq 1.93 \cdot \Delta^2$, for Δ large enough. In a different direction, Wang [103] has shown that $\chi(L(G)^2) \leq (4k-2)\Delta - 2k^2 + 1$ for all $k-$degenerate graphs.

Since the blown-up $5-$cycles contain many odd cycles, it is natural to ask how the conjecture changes if one restricts to bipartite graphs. Faudree, Gýarfás, Schelp and Tuza [44] conjectured that among bipartite graphs, the balanced complete bipartite graphs $K_{\Delta,\Delta}$ are extremal.

Conjecture 4.1.2 ([44])**.** *Let G be a bipartite graph of maximum degree Δ. Then*

$$\chi(L(G)^2) \leq \Delta^2.$$

More generally, for a bipartite graph on parts X_1 and X_2, one can define the left and right maximum degrees $\Delta_i := \max_{v \in X_i}(\deg(v))$, $i \in \{1,2\}$. It is believed that $\chi(L(G)^2 \leq \Delta_1\Delta_2$ and this is confirmed [88, 62] for all $\Delta_1 \leq 3$.

Rather than excluding all odd cycles, one can instead remove an even cycle. Since graphs without a fixed even cycle have a lower Turán number than bipartite graphs (cf. e.g. [48]), one may expect a stronger effect on the maximal value of the strong chromatic index as well. Indeed, Mahdian [82] has shown that for every C_4-free graph, $\chi(L(G)^2) \leq (2+\epsilon)\Delta^2/\log(\Delta)$, thus exhibiting a reduction in the strong chromatic index by a logarithmic factor. Subsequently Vu extended this result to any fixed bipartite graph H, showing that there is a constant $K_H > 0$ such that for any $H-$free graph G of maximum degree Δ, the strong chromatic index is bounded by $K_H \cdot \Delta^2/\log(\Delta)$. Up to the constant, this is sharp.

In this chapter we take a step back. Instead of $\chi(L(G)^2)$ we investigate the *strong clique number* $\omega(L(G)^2)$, the clique number of the square of the line graph. As opposed to the chromatic number, determining the clique number is a local problem and thus one expects it to be more tractable. At the same time, considering this more modest problem may provide clues on how (not) to tackle its chromatic counterpart. Although in general the clique number is merely a lower bound for the chromatic number, there are major graph classes $\mathcal{G}$ for which it has been shown or conjectured that $\sup_{G\in\mathcal{G}} \chi(L(G)^2) = \sup_{G\in\mathcal{G}} \omega(L(G)^2)$. In particular this is believed to be true for the class of all graphs of maximum degree Δ and the class of all bipartite graphs of maximum degree Δ, as well as the class of all C_5-free graphs of maximum degree Δ.

In this chapter we show that $\omega(L(G)^2) \leq 5/4\Delta^2$ for all triangle-free multigraphs of maximum degree Δ. This is sharp for even Δ (and almost sharp for odd Δ) and it equals the conjectured best possible bound for $\chi(L(G)^2)$. On the other hand, we show that $\sup_{G\in\mathcal{G}}\omega(L(G)^2)$ is of markedly smaller order than $\sup_{G\in\mathcal{G}}\chi(L(G)^2)$ if $\mathcal{G}$ is the class of (bipartite) C_4-free graphs of maximum degree Δ, or the class of (bipartite) graphs of maximum degree Δ witout cycles of order $\in\{k,k+1,k+2\}$, (for some $k\geq 4$).

For bipartite graphs of maximum degree Δ, the extremal value of $\omega(L(G)^2)$ has been determined in a classic paper by Faudree, Gýarfás, Schelp and Tuza [44]. It is attained by the complete balanced bipartite graph $K_{\Delta,\Delta}$.

Theorem 4.1.3 ([44]). *Let G be a bipartite graph of maximum degree Δ. Then*

$$\omega(L(G)^2)\leq\Delta^2.$$

Faron and Postle [42] have derived a strengthening in terms of the *Ore-degree* of G, which is defined as $\sigma(G) := \max_{xy\in E(G)}(\deg(x)+\deg(y))$. They showed that for all bipartite multigraphs G, $\omega(L(G)^2)\leq\frac{1}{4}\sigma(G)^2$. We prove a strengthening of this; see Theorem 4.1.6. In particular this demonstrates that for bipartite graphs, the extremal value of $\omega(L(G)^2)$ corresponds to the conjectured extremal value of $\chi(L(G)^2)$. The same correspondence is expected to hold for the class of all graphs. One piece of evidence for this is that the conjectured extremal graphs for the colouring problem (the blown-up $5-$cycles) each form a clique in the square of their line graph.

Conjecture 4.1.4 ([44]). *Let G be a graph of maximum degree Δ. Then*

$$\omega(L(G)^2)\leq\frac{5}{4}\Delta^2.$$

Recently, Śleszyńska-Nowak [99] has shown that $\omega(L(G)^2)\leq\frac{3}{2}\Delta(G)^2$. Subsequently, Faron and Postle [42] improved this to $\omega(L(G)^2)\leq\frac{4}{3}\Delta(G)^2$. Moreover, they gave a conditional proof of Conjecture 4.1.4, under the additional condition that whenever the edges of a subgraph $H\subseteq G$ form a clique in $L(G)^2$, it holds for all proper bipartite subgraphs H^* of H that

$$|E(H^*)|\leq\frac{1}{4}\cdot\left(\max_{xy\in E(H^*)}\deg_{G[V(H^*)]}(x)+\deg_{G[V(H^*)]}(y)\right)^2.$$

4.1.2 New results for excluded cycles and paths

We take another direction, focusing on how the strong clique number behaves if we exclude cycles or paths as a subgraph. First, we show that Conjecture 4.1.4 is true for all *triangle-free* graphs. This is sharp because the blown-up $5-$cycles are triangle-free.

Theorem 4.1.5. *If G is a triangle-free graph of maximum degree Δ, then*

$$\omega(L(G)^2) \le \tfrac{5}{4}\Delta^2.$$

Now let us see what happens if instead of a triangle we exclude a larger odd cycle (and we take the number of vertices large enough). It turns out that in terms of bounding $\omega(L(G)^2)$, this reduces to the case of bipartite graphs. In particular:

Theorem 4.1.6. *If G is a C_5-free multigraph of maximum degree Δ and Ore-degree σ, then*

$$\omega(L(G)^2) \le \frac{\sigma^2}{4} \le \Delta^2.$$

This result extends a result proved by Mahdian [82] for C_5-free graphs without an induced matching of size 2. Intuitively, excluding an even cycle rather than an odd cycle should have a greater effect, since it forces the graph to be more sparse. Indeed we show the following.

Theorem 4.1.7. *If G is a C_4-free graph with maximum degree $\Delta > 3$, then*

$$\omega(L(G)^2) \le 3(\Delta - 1).$$

This is sharp.

Thus the maximum possible value of $\omega(L(G)^2)$ among C_4-free graphs is linear in Δ. On the other hand, by analyzing a suitable random graph it can be shown (see [82]) that for all $g \ge 4$ and sufficiently large Δ, there exists a graph of girth at least g and maximum degree Δ such that $\chi(L(G)^2) \ge \left(\frac{1}{2} + o(1)\right)\Delta^2/\ln(\Delta)$. In particular we can conclude that excluding C_4's has a much stronger diminishing effect on $\omega(L(G)^2)$ than it has on $\chi(L(G)^2)$.

An extremal graph for Theorem 4.1.7 is a triangle $x_1x_2x_3$ of which each vertex has $\Delta - 2$ additional neighbours of degree 1 outside the triangle. We call this type of graph a *hairy clique*. More generally, for an integer $k \ge 1$, we define the hairy clique H_k as the graph consisting of a clique on k vertices, each of which has $\Delta - k + 1$ neighbours of degree 1 outside the clique. Note that all edges of H_k are within distance two, so $\omega(L(H_k)^2) = |E(H_k)| = k\left(\Delta - \frac{k-1}{2}\right)$.

We conjecture the following generalization of Theorem 4.1.7. Here P_k denotes the path on k vertices.

Conjecture 4.1.8. *Let $k \ge 1$ be an integer. Let G be a graph of maximum degree $\Delta \ge \max(4, k-1)$. Suppose that either G is $P_{k+3}-$free, or $k+1$ is even and G is C_{k+1}-free. Then*

$$\omega(L(G)^2) \le \omega(L(H_k)^2) = k\left(\Delta - \tfrac{k-1}{2}\right).$$

We have the following additional evidence, which is sharp up to a small term only depending on k for the path-free result.

Theorem 4.1.9. *Let G be a graph with maximum degree $\Delta \ge 2$.*

- *Let $k \ge 1$ be an integer. If G is $P_{k+3}-$free then*

$$\omega(L(G)^2) \le k \cdot (\Delta - 1) + 2.$$

- *Let $l \geq 3$ be an integer. If G does not contain any cycle of order $\in \{l+1, l+2, l+3\}$, then*
$$\omega(L(G)^2) \leq l \cdot (\Delta - 1) + 2.$$

As mentioned before, $\sup_{G \in \mathcal{G}} \chi(L(G)^2)$ is of the order $\Delta^2/\log(\Delta)$ for $\mathcal{G}$ the class of C_{2k}−free graphs. So again we observe a marked difference with $\sup_{G \in \mathcal{G}} \omega(L(G)^2)$, which is only of linear order in Δ.

Note that for the cycle-free result in Theorem 4.1.9 we need to exclude more than one cycle, namely cycles of three consecutive orders (yet if we choose l even then only one of those cycles is of even order!). This is an artefact of the proof. In the proof we argue by contradiction that $\omega(L(G)^2)$ is large. We take a maximum clique in $L(G)^2$ and derive the existence of certain long paths that start and end on edges from this clique. Two such edges need to be within distance 2, so there must be a long cycle. Thus the cycle-free result follows as a relatively simple corollary to the path-free result.

In summary, so far we have described the effect on $\omega(L(G)^2)$ if we exclude all odd cycles (ie: bipartite graphs), a triangle, a larger odd cycle, a path or an even cycle. We now zoom in one step further and ask what happens if we exclude all odd cycles *and* an even cycle. For comparison, we first want to know how small $\sup_{G \in \mathcal{G}} \chi(L(G)^2)$ can get for $\mathcal{G}$ the class of bipartite graphs of girth at least $g \geq 4$. We minimally adapt an argument from [82] to demonstrate that even in this very restricted graph class, $\chi(L(G)^2)$ can be as high as $\Omega(\Delta^2/\log(\Delta))$.

Lemma 4.1.10. *For every $g \geq 4$ and sufficiently large Δ, there is a bipartite graph G of girth at least g and maximum degree Δ, such that*
$$\chi(L(G)^2) \geq (\tfrac{1}{2} - o(1)) \frac{\Delta^2}{\ln(\Delta)}.$$

As we saw in Theorem 4.1.9, $\omega(L(G)^2)$ can be at most linear in Δ when three cycles of consecutive order are excluded. We conjecture and partially prove that this extremal value is reduced by roughly a factor 2 if we additionally impose that the graphs be bipartite.

Conjecture 4.1.11. *Let $k \geq 2$ be an integer. Let G be a C_{2k}-free bipartite graph of maximum degree Δ. Then*
$$\omega(L(G)^2) \leq k\Delta + 1 - k.$$
If true, this bound is sharp for $k \leq \Delta + 1$. An extremal graph would be the complete bipartite graph on parts X and Y of sizes $k-1$ respectively Δ, where one of the vertices in Y has $\Delta - k + 1$ additional neighbours of degree 1.

We furthermore believe that if G is P_{2k+1}-free, then
$$\omega(L(G)^2) \leq \begin{cases} k\Delta + 1 - k & \text{if } k \neq \Delta; \\ k\Delta & \text{if } k = \Delta. \end{cases} \tag{4.1}$$
with the same extremal graphs except in the case $k = \Delta$, for which the complete balanced bipartite graph $K_{\Delta,\Delta}$ should be extremal. We now present some evidence for Conjecture 4.1.11 and the proposed bound in (4.1).

Lemma 4.1.12. *Conjecture 4.1.11 and bound (4.1) are true for* $k = 2$ *and* $k \geq \Delta + 1$*.*

For general k we obtain the following result, which is very close to sharp in the regime $k \leq \Delta/2 + 1$.

Theorem 4.1.13. *Let* $k \geq 2$ *be an integer. If* G *is a bipartite graph of maximum degree* Δ *that is* $P_{2k+1}-$*free or* ($C_{2k}-$*free and* $C_{2k+2}-$*free), then*

$$\omega(L(G)^2) \leq \max(k\Delta, 2k(k-1)).$$

As it turns out, any upperbound in the bipartite graph setting also yields a bound for more general classes of graphs that are both C_3-free and C_5-free.

Lemma 4.1.14. *Let* $\mathcal{G}$ *be a class of graphs that is* C_3-*free,* C_5-*free and invariant under vertex-deletion. Let* $\mathcal{G}_{bip}$ *be the maximal subclass of* $\mathcal{G}$ *containing only bipartite graphs. Then*

$$\max_{G \in \mathcal{G}} \omega(L(G)^2) = \max_{G \in \mathcal{G}_{bip}} \omega(L(G)^2).$$

Proof. Since $\mathcal{G}_{\text{bip}} \subseteq \mathcal{G}$, it suffices to show that $\max_{G \in \mathcal{G}} \omega(L(G)^2) \leq \max_{G \in \mathcal{G}_{\text{bip}}} \omega(L(G)^2)$. Let $G \in \mathbb{G}$ be such that $\omega(L(G)^2)$ is maximum. Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$. Let $e = uv \in E(H)$. Because G is C_3-free and C_5-free, it follows that $G^* := G[N(u) \cup N(v) \cup N(N(u)) \cup N(N(v))]$ is bipartite. Since H is a subgraph of G^* and all possible edges (in G) between edges of H are contained in G^*, it follows that $\omega(L(G^*)^2) = |E(H)| = \omega(L(G)^2)$. □

In particular, for any graph H, we can take $\mathcal{G}$ to be the class of graphs of maximum degree Δ that are C_3-, C_5- and $H-$free. Lemma 4.1.14 says that in order to maximize the strong clique number for this class, it suffices to consider $H-$free bipartite graphs. This yields the following (almost sharp) corollary.

Corollary 4.1.15. *Let* $2 \leq k \leq \frac{\Delta}{2} + 1$*. If* G *is a graph of maximum degree* Δ *without any cycle of order* $\in \{3, 5, 2k, 2k+2\}$*, then*

$$\omega(L(G)^2) \leq k\Delta.$$

4.1.3 New results for bounded Hadwiger number

Rather than excluding paths and cycles as a subgraph, as we did up to now, one could also ask how $\omega(L(G)^2)$ behaves if we forbid a minor. In this subsection we upperbound $\omega(L(G)^2) \leq h(G) \cdot \Delta(G)$, where $h(G)$ is the Hadwiger number of G, the order of the largest clique that is a minor of G. This is close to sharp because of the hairy cliques. Assuming Hadwiger's conjecture, we derive that essentially the same upper bound should hold for the strong chromatic index.

Theorem 4.1.16. *Let* $1 \leq k \leq \frac{\Delta+1}{2}$*. Let* G *be a graph of maximum degree* Δ *and without* K_k *as a minor. Then*

$$\omega(L(G)^2) \leq (k-1) \cdot \Delta.$$

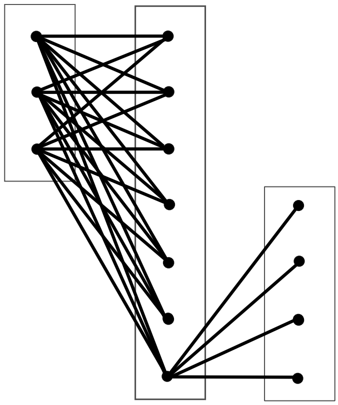

Figure 4.1: The complete bipartite graph $K_{k-1,\Delta}$, where one vertex in the part of size Δ has $\Delta - k + 1$ additional neighbours. This is a $P_{2k+1}-$free and $C_{2k}-$free graph that (for $k \neq \Delta$) attains the bound of Conjecture 4.1.11.

Albeit the bound given in Theorem 4.1.16 is close to sharp in the most relevant regime $k \ll \Delta$, there is still some room to improve it. The mild condition $1 \leq k \leq \frac{\Delta+1}{2}$ is an artefact of the proof and in fact, it is easy to show along the same lines that $\omega(L(G)^2) \leq (k-1)(2\Delta - 1)$, unconditional on the ratio of k and Δ. We believe that hairy cliques are the extremal graphs for this problem, just as for the class of graphs without a given path or even cycle.

Conjecture 4.1.17. *Let $k \in \mathbb{N}_{\geq 1}$. Let G be a graph without K_k as a minor and of sufficiently large maximum degree Δ. Then*

$$\omega(L(G)^2) \leq (k-1) \cdot (\Delta - \tfrac{k-2}{2}).$$

If true, this bound is attained by hairy cliques.

If we impose additionally that *all* edges of G are within distance two, so that $L(G)^2$ is a complete graph, then we also have a short argument for the strong clique number of *multigraphs*. Using the Tutte-Berge formula [10], the following result can be derived.

Lemma 4.1.18. *Let $k \in \mathbb{N}_{\geq 1}$. Let G be a multigraph of maximum degree Δ without K_k as a minor. If $L(G)^2$ is a complete graph, then*

$$\omega(L(G)^2) = |E(G)| \leq (k - \tfrac{1}{2}) \cdot \Delta.$$

For $k = 5$ this is sharp up to a constant, because there exists a planar multigraph with $\omega(L(G)^2) = \frac{9}{2}\Delta - 6$, namely the multigraph obtained from the octahedral graph on vertices $abcdef$ by blowing up the edges in the triangle abc to edges of mulitiplicity $\Delta/2 - 1$ and adding three extra edges of mulitiplicity $\Delta - 4$ incident to d, e and f. In particular this implies that the extremal values of the strong clique number for graphs respectively multigraphs (with bounded Hadwiger number and bounded maximum degree) do not coincide. This contrasts with e.g. Theorem 4.1.6.

If we assume Hadwiger's conjecture is true, then we can extend the above results to the strong chromatic index.

Lemma 4.1.19. *Let $k \in \mathbb{N}_{\geq 1}$. Let G (respectively M) be a graph (respecitvely multigraph) of maximum degree Δ without K_k as a minor. If Hadwiger's conjecture holds true, then*

$$\chi(L(G)^2) \leq (k-1) \cdot (\Delta + 1) \text{ and}$$
$$\chi(L(M)^2) \leq (k-1) \cdot \frac{3}{2}\Delta.$$

Proof. By Vizing's theorem, the edges of G can be coloured with at most $\Delta+1$ colours. Let C be one of the colour classes. Contract each edge $e \in C$ to a vertex $v(e)$, yielding a new graph G^*. Hadwiger's conjecture implies that we can vertex-colour G^* with at most $h(G^*) \leq h(G)$ colours. Now we give each $e \in C$ the colour of its contracted vertex $v(e)$. This yields a colouring of C such that each edge pair (in C) at distance 2 (with respect to G) has different colours. Doing this for each of the $\Delta + 1$ colour classes, we obtain a strong edge-colouring with at most $h(G) \cdot (\Delta + 1)$ colours. This proves the result for graphs. For multigraphs, apply the same argument with Shannon's theorem instead of Vizing's theorem. □

4.1.4 From clique number to fractional chromatic number

Before going to the remaining proofs, we wish to illustrate how knowledge of the clique number sometimes can provide good upperbounds on the (fractional) chromatic number. Reed conjectured that for all graphs G, its chromatic number is essentially bounded from above by the average of its clique number and maximum degree.

Conjecture 4.1.20 (Reed [91])**.** *For any graph G,*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

This conjecture has been validated for several classes of graphs, among which are line graphs [71] and even more generally claw-free graphs [70],[72], but remains wide open in general. Molloy and Reed [87] proved a *fractional* analogue of Reed's conjecture for all graphs.

A fractional vertex $c-$colouring of a graph G can be described as a collection $\{S_1, \ldots, S_l\}$ of stable sets with associated nonnegative weights $\{w_1, \ldots, w_l\}$ such that for every vertex v, $\sum_{S_i : v \in S} w_i = 1$ and $\sum_{i=1}^{l} w_i = c$. Note that if we would also require all weights to be 1 then we would obtain the definition of an ordinary proper colouring. The *fractional chromatic number* $\chi_f(G)$ of G is the smallest c for which G has a fractional vertex $c-$colouring.

Theorem 4.1.21 (Molloy and Reed [87])**.** *For any graph G,*

$$\chi_f(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

Since $\Delta(L(G)^2) \leq 2\Delta(G)^2 - 2\Delta(G) + 1$ for all graphs, we obtain as a corollary from Theorem 4.1.5 that:

Corollary 4.1.22. *If G is a triangle-free graph of maximum degree Δ, then*

$$\chi_f(L(G)^2) \leq \tfrac{13}{8}\Delta^2 - \Delta + 1.$$

Furthermore, if Reed's conjecture holds true then this bound (rounded upwards) holds for $\chi(L(G)^2)$ as well.

This concludes the introduction. We will now provide the remaining proofs of our results.

4.2 Triangle-free graphs.

In this section, we prove Theorem 4.1.5. For a (multi)graph G and a sub(multi)graph H, the Ore-degree of H in G is defined as $\sigma_G(H) := \max_{xy \in E(H)} (\deg_G(x) + \deg_G(y))$, where $\deg_G(x)$ denotes the degree of x in G. The following lemma of Faron and Postle plays a central role in the proof.

Lemma 4.2.1 (Faron and Postle [42])**.** *If G is a bipartite multigraph and H is a sub(multi)graph of G such that $E(H)$ is a clique in $L(G)^2$, then $|E(H)| \leq \Delta(H) \cdot (\sigma_G(H) - \Delta(H)) \leq \frac{\sigma_G(H)^2}{4}$.*

We will apply this lemma to an appropriate bipartite subgraph of our triangle-free graph.

Proof of Theorem 4.1.5
Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$. From now on we call H and its edges *blue*. Let $v \in V(G)$ be of maximal blue degree s. Let V_T denote the set of vertices that are incident to an edge of H that is not incident to $N[v]$. Let $G_T = (V_T, E_T)$ be the graph induced by V_T and let $H_T = (V_T, E_T \cap E(H))$ be the blue subgraph of G_T.

Let $C_1, C_2, \ldots$ denote the connected components of H_T. Let pq be an edge in component C_i. For all $x \in N_H(v)$, the blue edges xv and pq must be within distance 2. They are not incident, so either $xp \in E(G)$ or $xq \in E(G)$. By triangle-freeness, we cannot have both $xp, xq \in E(G)$. It follows that *pq partitions* $N_H(v)$ into $A_i := N_G(p) \cap N_H(v)$ and $\overline{A_i} := N_G(q) \cap N_H(v)$. We will call $(A_i, \overline{A_i})$ the partition induced by pq. Now suppose C_i contains another edge qr which is incident to pq. Then by triangle-freeness, qr induces the same partition. It follows inductively that all edges in C_i induce the *same* partition $(A_i, \overline{A_i})$ of $N_H(v)$.

Let $C_1, \ldots, C_k$ be the components that induce the trivial partition $(\emptyset, N_H(v))$ (if they exist). Let $M = |C_1| + \ldots + |C_k|$ denote the number of (blue) edges that are in these 'trivial' components. On the other hand, let $G_B := G[\bigcup_{i \geq k+1} V(C_i)]$ and $H_B := H[\bigcup_{i \geq k+1} V(C_i)]$ be the graphs induced by the remaining 'nontrivial' components.

Claim 1 $M \leq (\Delta - s)\Delta$.

Claim 2 $\sigma_{G_B}(H_B) \leq 2\Delta - s - \frac{M}{\Delta}$.

Claim 3 G_B is bipartite.

We postpone the proofs of these claims. Note that $E(H_B)$ is not only a clique in $L(G)^2$ but also in $L(G_B)^2$. So by Claim 3, we may apply Lemma 4.2.1 and then Claim 2, yielding

$$|E(H_B)| \leq \frac{\sigma_{G_B}(H_B)^2}{4} \leq \frac{(2\Delta - s - \frac{M}{\Delta})^2}{4}.$$

It follows that $\omega(L(G)^2)$ is at most

$$\begin{aligned} |E(H)| &= \#\{e \in E(H) \mid e \text{ incident to } N_G(v)\} + |E(H_T)| \\ &\leq s \cdot |N_G(v)| + M + |E(H_B)| \\ &\leq s\Delta + M + \frac{(2\Delta - s - \frac{M}{\Delta})^2}{4} \\ &= \Delta^2 + \frac{1}{4}\left(s + \frac{M}{\Delta}\right)^2 \leq \tfrac{5}{4}\Delta^2, \end{aligned}$$

where we used Claim 1 in the last line. This concludes the proof, conditioned on the validity of Claims 1, 2 and 3.

Given the $i-$th component C_i, let X_i respectively Y_i denote the set of vertices in C_i whose neighbourhood in $N_H(v)$ is A_i respectively $\overline{A_i}$. Note that X_i is complete to A_i and Y_i is complete to $\overline{A_i}$. Furthermore, the bipartite subgraph of H induced by C_i has parts X_i and Y_i.

Proof of Claim 1
If C_i is a trivial component ($1 \leq i \leq k$) then Y_i is complete to $\overline{A_i} = N_H(v)$. Therefore $|\bigcup_{1 \leq i \leq k} Y_i| \leq \Delta$, and for the same reason all $y \in \bigcup_{1 \leq i \leq k} Y_i$ satisfy $|N_{H_T}(y)| \leq \Delta - s$. So $M \leq |N_{H_T}\left(\bigcup_{1 \leq i \leq k} Y_i\right)| \leq \Delta(\Delta - s)$.

Proof of Claim 2
Let $e = pq \in E(H_B)$. Then for all $x \in N_H(v)$, x must be adjacent to either p or q. So there are $|N_H(v)| = s$ edges between $\{p, q\}$ and $N_H(v)$. Also, pq must be at distance 2 of every of the M edges induced by the trivial components. So there are at least $\frac{M}{\Delta}$ edges between $\{p, q\}$ and the trivial components. So at least $s + \frac{M}{\Delta}$ edges incident to $\{p, q\}$ are not in G_B. It follows that $\sigma_{G_B}(e) = d_{G_B}(p) + d_{G_B}(q) \leq 2\Delta - s - \frac{M}{\Delta}$.

Proof of Claim 3
Suppose there are two different nontrivial components, C_i and C_j. We will first show that we may then assume that either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Indeed: if either $\overline{A_j} \subseteq A_i$ or $A_i \subseteq \overline{A_j}$, then after interchanging X_j and Y_j (and thus interchanging A_j and $\overline{A_j}$), we obtain $A_j \subseteq A_i$ or $A_i \subseteq A_i$. So we may assume for a contradiction that none of $A_i \subseteq A_j, A_i \subseteq \overline{A_j}, A_j \subseteq A_i, \overline{A_j} \subseteq A_i$ holds. But then there exist $a \in A_i \cap \overline{A_j}, b \in A_i \cap A_j, c \in \overline{A_i} \cap A_j$ and $d \in \overline{A_i} \cap \overline{A_j}$. Furthermore, because each component contains at least one blue edge, there are blue edges $(x_i, y_i) \in (X_i \times Y_i)$ and $(x_j, y_j) \in (X_j \times Y_j)$ that have to be connected by an edge in order to have them within distance 2. If $x_i x_j$ is an edge, then $x_i x_j b$ forms a triangle. Similarly, if $x_i y_j$, $y_i y_j$ or $x_j y_i$ is an edge then $x_i y_j a$, $y_i y_j d$ or $x_j y_i c$ is a triangle, respectively. Contradiction.

It follows that we can reorder the components by inclusion, so that $A_{k+1} \subseteq A_{k+2} \subseteq \ldots$. Now we are ready to show that G_B is bipartite, on parts $X := \bigcup_{i \geq k+1} X_i$ and $Y := \bigcup_{i \geq k+1} Y_i$. Suppose X is not a stable set. Then there are $x_1, x_2 \in X$ that form

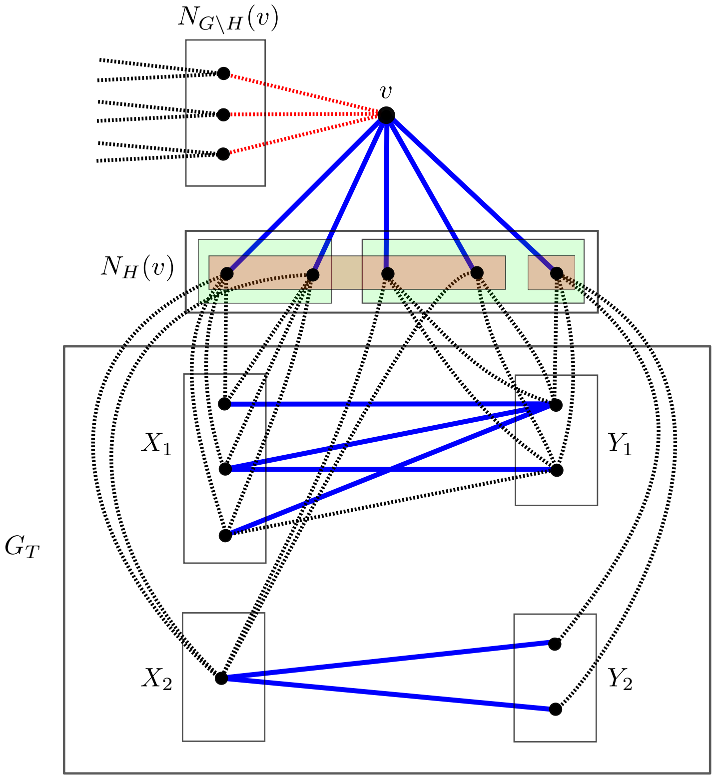

Figure 4.2: The structure described in Theorem 4.1.5. Blue edges are in H, red edges are not in H and black edges could be either. In this picture, $H_T \subseteq G_T$ has two (blue) connected components, induced by $X_1 \cup Y_1$ respectively $X_2 \cup Y_2$. The blue neighbourhood $N_H(v)$ is partitioned into two sets A_1 (its left two vertices) and $\overline{A_1}$ (the remaining four vertices on the right), such that X_1 is complete to A_1 and Y_1 is complete to $\overline{A_1}$. The neighbourhoods of X_2 and Y_2 induce another partition of $N_H(v)$. *Not all* edges are depicted here. In particular, we have left out the (possibly red) edges inside G_T that ensure that all of its blue edges are within distance 2.

an edge, where $x_1 \in X_i$ and $x_2 \in X_j$ for some $i \leq j$. Since $\emptyset \neq A_i \subseteq A_j$, there must be a triangle in $x_1x_2A_i$. Contradiction. Similarly, suppose Y is not a stable set. Then there are $y_1, y_2 \in Y$ that form an edge, where $y_1 \in Y_i$ and $y_2 \in Y_j$ for some $i \leq j$. Since $\emptyset \neq \overline{A_j} \subseteq \overline{A_i}$, there must be a triangle in $y_1y_2\overline{A_j}$. Contradiction. □

4.3 Multigraphs without C_5

In this section we prove Theorem 4.1.6. We actually prove the following slightly stronger lemma, which generalizes Lemma 4.2.1.

Lemma 4.3.1. *If G is a C_5-free multigraph and H is a sub(multi)graph of G such that $E(H)$ is a clique in $L(G)^2$, then $|E(H)| \leq \Delta(H) \cdot (\sigma_G(H) - \Delta(H)) \leq \frac{\sigma_G(H)^2}{4}$.*

Proof of Lemma 4.3.1.
Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$. Let $v \in V(G)$ be a vertex that maximizes $|N_H(v)|$. Note that since H is a multigraph, $\Delta(H)$ may be strictly larger than $|N_H(v)|$.

We may assume that $|N_H(v)| \geq 2$. Indeed, if $|N_H(v)| = 1$ then we can fix an edge $uv \in E(H)$ of maximum Ore-degree $\sigma_G(H)$. Note that the multiplicity of this edge is equal to $\Delta(H)$. Each vertex in $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$ is incident to at most $\Delta(H)$ edges of H, and there are at most $\sigma_G(H) - 2\Delta(H)$ such vertices. Therefore $|E(H)| \leq \Delta(H) + (\sigma_G(H) - 2\Delta(H)) \cdot \Delta(H) \leq \Delta(H) \cdot (\sigma_G(H) - \Delta(H))$, as desired.

Now let $E^* \subseteq E(H)$ denote the set of those edges $st \in E(H)$ for which $s, t \notin N_G(v)$. Let $st \in E^*$. Then for all $u \in N_H(v)$, vu must be within distance 2 of st, so either $us \in E(G)$ or $ut \in E(G)$. Without loss of generality, $us \in E(G)$. Because G has no C_5 and $|N_H(v)| \geq 2$, it follows that t is anticomplete to $N_H(v) \setminus \{u\}$, so in fact s is complete to $N_H(v)$ and t is anticomplete to $N_H(v)$. We derived this for all $st \in E^*$, so there exists a subset $S \in V(H)$ such that

(i) each edge in E^* has an endpoint in S and

(ii) S is complete to $N_H(v)$.

Since each edge of H is either in E^* (and thus has an endpoint in S) or has an endpoint in $N_G(v)$, we can cover $E(H)$ with the following three subsets:

$$E_S := \{e \in E(H) \mid e \text{ has an endpoint in } S\},$$

$$E_1 := \{e \in E(H) \mid e \text{ has an endpoint in } N_G(v) \setminus N_H(v)\}, \text{ and}$$

$$E_2 := \{e \in E(H) \mid e \text{ has an endpoint in } N_H(v) \text{ but not in } S\}.$$

Each vertex is incident to at most $\Delta(H)$ edges of H, so $|E_S| \leq \Delta(H) \cdot |S|$ and $|E_1| \leq \Delta(H) \cdot |N_G(v) \setminus N_H(v)| \leq \Delta(H) \cdot (\deg_G(v) - \Delta(H))$. To bound E_2 we need that, by property (ii), each vertex $x \in N_H(v)$ is incident to at most $\deg_G(x) - |S|$ edges that are not incident to S. Thus, $|E_2| \leq \sum_{x \in N_H(v)} (\deg_G(x) - |S|) \leq \Delta(H) \cdot (\sigma_G(H) - \deg_G(v) - |S|)$. In conclusion,

$$|E(H)| \leq |E_S| + |E_1| + |E_2| \leq \Delta(H) \cdot (\sigma_G(H) - \Delta(H)).$$

Note that the right hand side is maximized for $\Delta(H) = \lceil \sigma_G(H)/2 \rceil$, so $|E(H)| \leq \sigma^2_{G(H)}/4$. □

4.4 Without an even cycle

4.4.1 Without C_4

In this section we prove Theorem 4.1.7. We proceed by case analysis. In subcases 2.1.2, 2.2.1 and 2.2.2.2 we can reduce to the case of the neighbourhood of a triangle, which constitutes exactly the extremal hairy clique. In the other cases, we derive bounds that are of smaller order 2Δ.

Proof of Theorem 4.1.7.

We will use the following notation. Let H be a *blue* subgraph of G whose edges form a maximum clique in $L(G)^2$. Thus, bounding $\omega(L(G)^2)$ is equivalent to bounding $|E(H)|$. In this subsection, we write $N(x)$ for the neighbourhood of a vertex x with respect to G. Choose an edge $e := uv$ in H. We define the following vertex subsets: $A_u := N(u)\setminus\{v\}$ and $A_v := N(v)\setminus(\{u\} \cup N(u))$. For short we will write $A := A_u \cup A_v$ for the neighbourhood of uv and we also need the second-order neighbourhood $B := N(A)\setminus(A \cup \{u, v\})$.

For any disjoint $X, Y \subset V(G)$, we write $E_X := E(H[X])$ and we write $E_{X,Y}$ for the set of edges in H between X and Y. Similarly, for a vertex x and a set Y, we write $E_{x,Y}$ for the set of edges in H between x and Y. For vertices $x, y \in V(G)$, y is said to be a *blue neighbour* of x if $xy \in E(H)$.

Note that the edges of H can be decomposed as a disjoint union, as follows.

$$E_{V(H)} = E_{u,A} \cup E_{v,A} \cup E_{A_u,B} \cup E_{A_v,B} \cup E_A \cup \{e\}\,.$$

We will use the following claim a few times.

Claim. $|E_A| \le 1$.

Proof. If $|E_A| \ge 2$, then $G[A]$ must contain a 2−path, which forms a C_4 with u and/or v. □

We now start the case analysis.

Case 1 *No vertex in A has ≥ 2 blue neighbours in B.*

The first thing to notice is that A_u and A_v each contain at most 3 vertices with a blue edge to B. Indeed, if there are four such vertices $x_1, \ldots, x_4$ with blue neighbours $y_1, \ldots, y_4 \in B$ respectively, then the $(y_i)_{1\le i\le 4}$ must be pairwise distinct to prevent a C_4. Therefore the blue edges $(x_iy_i)_{1\le i\le 4}$ are pairwise at distance exactly 2. There can be at most 2 edges in $G[x_1, x_2, x_3, x_4]$ and these must be nonincident, for otherwise they form a C_4 with v. Say these edges are x_1x_2 and x_3x_4 (or a subset thereof). Then $y_1y_2y_2y_4$ is a C_4, contradiction. Second, it cannot be that both $|E_{v,A}| \ge 2$ and $|E_{A_u,B}| \ge 2$. Indeed, otherwise there are two vertices in $N_H(v)$ that must be complete to two vertices in $B \cap N_H(A_u)$, thus forming a C_4, contradiction. So $|E_{v,A}| + |E_{A_u,B}| \le \Delta$ (here we also use our assumption that $\Delta \ge 4$). And similarly, $|E_{u,A}| + |E_{A_v,B}| \le \Delta$. It follows that

$$\begin{aligned} |E_{V(H)}| &\le |E_{v,A}| + |E_{A_u,B}| + |E_{u,A}| + |E_{A_v,B}| + |E_A| + |\{uv\}| \\ &\le \Delta + \Delta + 1 + 1 = 2\Delta + 2. \end{aligned} \tag{4.2}$$

This is bounded from above by $3(\Delta - 1)$ if $\Delta \geq 5$. To conclude the same for the case $\Delta = 4$, we need to reduce the bound in equation (4.2) by 1.

If $|E_A| = 0$, then we get the desired improvement for free. If, on the other hand, E_A is nonempty, then its unique edge ab has both endpoints in either $N(u)\backslash N(v)$ or in $N(v)\backslash N(u)$, (otherwise $abuv$ would form a C_4). Wlog, assume ab is induced by $N(u)\backslash N(v)$. In that case it follows that $E_{v,A} = \emptyset$ (or otherwise a blue neighbour of v in A would have to be adjacent to a or b, forming a C_4.)

But then $|E_{v,A}| + |E_{A_u,B}| \leq 0 + \Delta - 1$, so we have again reduced the upper bound of equation (4.2) by 1, as desired.

Case 2 *At least one vertex in A has ≥ 2 blue edges to B. Wlog let $x \in A_u$ be such a vertex and let x_1^*, x_2^* denote two of its blue neighbours in B.*

- **Case** 2.1. *x is the only vertex in A_u that has a blue edge to B.*
  - **Case** 2.1.1. $vx \notin E(G)$.

    Suppose there exists $vy \in E_{v,A_v}$. Then $y \neq x$ because $vx \notin E(G)$. Also, $yx \notin E(G)$ because otherwise $uvyx$ would be a C_4. So y must be adjacent to the two blue neighbours x_1^*, x_2^* of x in B, in order to have vy within distance 2 of xx_1^* and xx_2^*. But then $xx_1^*x_2^*y$ forms a C_4. We conclude that

    $$E_{v,A_v} = \emptyset. \tag{4.3}$$

    We now show that it is impossible that both $|E_{u,A_u}| \geq 2$ and $|E_{A_v,B}| \geq 1$. Indeed, suppose there are $x_1, x_2 \in E_{u,A_u}$ and a blue neighbour $y^* \in B$ of some $y \in A_v$. Since $uy, uy^*, x_1y, x_2y \notin E(G)$ while yy^* must be within distance two of both ux_2 and ux_1, it follows that $y^*x_1, y^*x_2 \in E(G)$, yielding the four-cycle $ux_1x_2y^*$. Contradiction.

    If $|E_{A_v,B}| = 0$ then $|E_{V(H)}| \leq |N_H(u) \cup N_H(x)| + |E_A| \leq (2\Delta - 1) + 1 \leq 3(\Delta - 1)$, as desired. So we may from now on assume that

    $$|E_{u,A_u}| \leq 1. \tag{4.4}$$

    Next, we want to show that $|E_{A_v,B}| \leq 4$. Suppose for a contradiction that $|E_{A_v,B}| \geq 5$.

    Suppose first that there exists $y \in A_v$ with (at least) three blue neighbours y_1^*, y_2^*, y_3^* in B. Recall that $x_1^*, x_2^* \in B$ are two blue neighbours of x. Since $\{x_1^*, x_2^*\}$ has at most one element in common with $\{y_1^*, y_2^*, y_3^*\}$ (otherwise C_4) we may wlog assume that $\{x_1^*, x_2^*\} \cap \{y_1^*, y_2^*\} = \emptyset$. If $xy_1^*, xy_2^*, yx_1^*, yx_2^* \notin E(G)$, then $\{x_1^*, x_2^*\}$ must be complete to $\{y_1^*, y_2^*\}$, yielding a C_4. So wlog $xy_1^* \in E(G)$. This implies $xy_2^*, yx_1^*, yx_2^* \notin E(G)$ (otherwise there is a C_4 containing x and y). So in order to have yy_2^* within distance two of xx_1^* and xx_2^*, we must have $x_1^*y_2^*, x_2^*y_2^* \in E(G)$, yielding $xx_1^*y_2^*x_2^*$ as a C_4. Contradiction. So we have derived that each $y \in A_v$ has at most two blue neighbours in B.

Now suppose that some vertex $y_{12} \in A_v$ has two blue neighbours y_1^*, y_2^* in B. By the argument in the previous paragraph, we may exclude the possibilities $|\{x_1^*, x_2^*\} \cap \{y_1^*, y_2^*\}| \in \{0, 2\}$, so wlog $x_2^* = y_2^*$.

Additionally suppose there is another vertex $y_{34} \in A_v$ with two blue neighbours y_3^*, y_4^* in B. By the same argument, one of $\{y_3^*, y_4^*\}$ is equal to one of $\{x_1^*, x_2^*\}$. But $x_2^* = y_2^* \notin \{y_3^*, y_4^*\}$ (otherwise there is a C_4 containing y_2^* and v), so wlog $y_4^* = x_1^*$. Since we assumed that $|E_{A_v,B}| \geq 5$, there is yet another vertex $y_5 \in A_v$ with (at least) one neighbour $y_5^* \in B$. Since $y_5 y_5^*$ must be within distance two of xx_1^* and xx_2^* it follows that $xy_5^* \in E(G)$. Since $(G[A])$ does not contain a 2−path (otherwise C_4), at least one of $y_5 y_{12}, y_5 y_{34}$ is not an edge. Wlog $y_5 y_{12} \notin E(G)$. Then, in order to have $y_5 y_5^*$ within distance two of $y_{12} y_2^*$ and $y_{12} y_1^*$, we must either have $y_5^* y_{12} \in E(G)$ (in which case $y_5^* x y_2^* y_{12}$ is a C_4) or $y_5^* y_1^*, y_5^* y_2^* \in E(G)$ (in which case $y_5^* y_1^* y_{12} y_2^*$ is a C_4). Contradiction.

Thus we have derived that y_{12} is the only vertex in A_v with two blue neighbours in B (namely y_1^* and y_2^*). Since we assumed $|E_{A_v,B}| \geq 5$, there are three other vertices $y_3, y_4, y_5 \in A_v$ with unique blue neighbours $y_3^*, y_4^*, y_5^* \in B$, respectively. Since $G[A]$ does not contain a 2−path, the complement of the graph induced by $Y := \{y_{12}, y_3, y_4, y_5\}$ contains a C_4. This implies there is a C_4 in the graph induced by $\{y_1^*, y_2^*, y_3^*, y_4^*, y_5^*\}$, the set of blue neighbours of Y in B. Contradiction.

Thus, we have derived that no vertex in A_v has more than one blue neighbour in B. Now let $y_1, \ldots, y_5 \in A_v$ be vertices with respective unique blue neighbours $y_1^*, \ldots, y_5^* \in B$. Since $G[A]$ does not contain a 2−path, the complement of the graph induced by $\{y_1, y_2, y_3, y_4, y_5\}$ contains a C_4. This implies there is a C_4 in the graph induced by $\{y_1^*, y_2^*, y_3^*, y_4^*, y_5^*\}$. Contradiction.

This concludes our proof that

$$|E_{A_v,B}| \leq 4. \tag{4.5}$$

From (4.3), (4.4) and (4.5) it now follows that

$$\begin{aligned} |E_V(H)| &\leq |E_{u,A_u}| + |\{uv\}| + |E_{x,B}| + |E_{A_v,B}| + |E_A| \\ &\leq 1 + 1 + (\Delta - 1) + 4 = \Delta + 5 \\ &\leq 3(\Delta - 1). \end{aligned}$$

– **Case** 2.1.2. $vx \in E(G)$
Suppose there exists an edge $yy^* \in E_{A_v,B}$, with $y \in A_v$. Then absence of C_4−s dictates that y is not adjacent to x or any of its blue neighbours in B. Therefore y^* is adjacent to all blue neighbours of x in B, of which there are at least 2 by assumption. But then these neighbours form a C_4 with x and y^*. Contradiction. So $E_{A_v,B} = \emptyset$ and therefore all edges of H are incident to the triangle uxv. So $|E_{V(H)}| = |E(G[N_H(u) \cup N_H(x) \cup N_H(v)])| \leq 3(\Delta - 1)$.

- **Case** 2.2. *There is another vertex x_2 in A_u that has a blue edge to B.*
  Note that in this case $xx_2 \in E(G)$, for otherwise there would be a C_4 in the graph induced by u, x, x_2 and the blue neighbours of x and x_2 in B. Note furthermore that there cannot be a third vertex $x_3 \in A_u$ that has a blue edge to B, for otherwise the same argument yields $xx_3 \in E(G)$ so that x_2xx_3u would yield a C_4.

  - **Case** 2.2.1. $vx \notin E(G)$
    First, suppose there exists a blue edge $vy \in E_{v,A}$. Then $y \neq x$ (by assumption) and $y \neq x_2$ and $yx \notin E(G)$ (for otherwise $uvyx$ is a C_4). Since vy must be within distance two of the (blue) edges in $E_{x,B}$, it follows that y must be adjacent to both blue neighbours x_1^*, x_2^* of x in B. But then $xx_1^*x_2^*y$ forms a C_4. Contradiction. So we conclude that $E_{v,A} = \emptyset$. Second, suppose there is an edge $yy^* \in E_{A_v,B}$, with $y \in A_v$ and $y^* \in B$. Let z_1^*, z_2^* be two blue neighbours of x in B and let z_3^* be a blue neighbour of x_2 in B. Recall that $xx_2 \in E(G)$ and, as before, $y \notin \{x, x_2\}$ and $yx, yx_2 \notin E(G)$. So in order to have yy^* within distance 2 of xz_1^*, xz_2^* and $x_2z_3^*$, we must have for all $i \in \{1,2,3\}$ that either $y^*z_i^* \in E(G)$ or $y^* = z_i^*$, and y^* can be equal to only one of the z_i^*. If $y^* = z_3^*$ then $xz_1^*z_2^*y^*$ will form a C_4. On the other hand, if (wlog) $y^* = z_1^*$, then $xy * z_3^*x_2$ forms a C_4. Contradiction. We conclude that $E_{A_v,B}$ must be empty too. It follows that all edges of H are indicent to the triangle uxx_2, so $|E_{V(H)}| \leq 3(\Delta - 1)$.
  - **Case** 2.2.2. $vx \in E(G)$
    By the argument of case 2.1.2, $E_{A_v,B} = \emptyset$.
    - **Case** 2.2.2.1. $E_{v,A_v} \neq \emptyset$.
      Let $vy \in E_{v,A_v}$ and $x_2x_2^* \in E_{A_u,B}$. Since $x_2y, vx_2 \notin E(G)$ (otherwise $uvyx_2$ or uvx_2x is a C_4), we must have $yx_2^* \in E(G)$. This holds for all such pairs, so in order to prevent a C_4, we must have $|E_{v,A_v}| + |E_{x_2,B}| \leq 2$. So $|E_{V(H)}| \leq |E_{v,A_v}| + |E_{x_2,B}| + |N_H(x) \cup N_H(u)| + |E_{A_v,B}| \leq 2 + (2\Delta - 1) + 0 = 2\Delta + 1$. This is bounded from above by $3(\Delta - 1)$ if $\Delta \geq 4$, which holds in this subcase because x is adjacent to u, v, x_2 and its ≥ 2 neighbours in B.
    - **Case** 2.2.2.2. $E_{v,A_v} = \emptyset$.
      In this case all edges of H are incident to the triangle uxx_2, so $|E_{V(H)}| \leq 3(\Delta - 1)$.

□

4.4.2 Without three consecutive cycles

Proof of Theorem 4.1.9 If $\Delta = 1$ then $\omega(L(G)^2) = 1$ so we may assume that $\Delta \geq 2$. We may also assume that $\Delta \geq 3$, because otherwise G is a path or a cycle, or a vertex-disjoint union of such graphs. For all such graphs, $\omega(L(G)^2) \leq 5 \leq l \cdot (\Delta - 1) + 2$, since by assumption $l \geq 3$. If G is a fivecycle, then $k \geq 3$ and $\omega(L(G))^2 = 5$. If G is a four-cycle then $k \geq 2$ and $\omega(L(G)^2) = 4$. Otherwise $\omega(L(G)^2) \leq 3$. In all these cases, $\omega(L(G)^2) \leq k \cdot (\Delta - 1) + 2$

For $\Delta \geq 3$, we will start with the proof for forbidden paths, and then adapt the arguments slightly to derive the result for forbidden cycles. Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$.

Write $F = E(H)$. Note that $|F| = \omega(L(G)^2) > \Delta$, for otherwise the conclusion of the theorem is already satisfied. It follows that G contains a path $P_4 = x_1y_1x_2y_2$ that starts and ends on edges x_1y_1, x_2y_2 from F. Indeed, let e_1 and e_2 be edges of F. If they are not incident to eachother, then there must be an edge between them and we have obtained the desired P_4. So we may assume that all edges of F are pairwise incident and in particular we can write $e_1 = xy$ and $e_2 = yz$. At most Δ edges meet in y, so F contains an edge e_3 that is not incident to y and therefore e_3 is incident to x. If $e_3 = xq \neq xz$ then $qxyz$ forms the desired P_4. Otherwise xyz forms a triangle of edges from F. Since $|F| \geq \Delta + 1 \geq 4$, there is a fourth edge in F incident to the triangle, again yielding a P_4.

Now define $F_1 = F \setminus (\{\text{ edges incident to } y_1 \text{ or } x_2\} \cup x_1y_2)$, a path $W_1 = y_1x_2$ and a longer 'preliminary' path $W_1^* = x_1y_1x_2y_2$, which is the P_4 whose existence we derived above.

In general, after the $i-$th iteration of the algorithm described below, we will obtain a path $W_i = y_1 \dots x_i$ and an extended preliminary path $W_i^* := x_1W_iy_i$ whose first and final edges x_1y_1, x_iy_i are elements of F. We also have a set F_i of unused-but-still-useful clique-edges, which we will set equal to $F \setminus (\{\text{ edges incident to } W_i\} \cup x_1y_i)$. The construction above shows this is true for $i = 1$. To show it is true in the $(i+1)$-th iteration ($i \geq 1$), we proceed as follows.

If F_i is nonempty, choose an edge $e_{i+1} = x_{i+1}y_{i+1} \in F_i$.

- **Case 1.** Suppose e_{i+1} is incident to either the first vertex (x_1) or the last vertex (y_i) of W_i^*. Assume without loss of generality that it is incident to y_i, and that $y_i = x_{i+1}$. Then we add e_{i+1} to our preliminary path; we set $W_{i+1} = W_ix_{i+1}$ and $W_{i+1}^* = W_ix_{i+1}y_{i+1}$. Note that (by construction of F_i) y_i is the only vertex in W_i^* that is incident to e_{i+1}, so W_{i+1}^* is a path as well.

- **Case 2.** Suppose that Case 1 does not apply, so that e_{i+1} is not incident to x_1 or y_i. Since $e_{i+1} \in F_i$, it follows that e_{i+1} is not incident to x_iy_i. Therefore there must be an edge e^* between e_{i+1} and x_iy_i. Wlog, e^* is incident to x_{i+1}, so we have $x_ix_{i+1} \in E(G)$ or $y_ix_{i+1} \in E(G)$.

  - **Case 2.1.** If $x_ix_{i+1} \in E(G)$. Then we again set $W_{i+1} = W_ix_{i+1}$ and $W_{i+1}^* = W_ix_{i+1}y_{i+1}$. Note that $W_{i+1} \neq W_i^*$.
  - **Case 2.2.** Else $y_ix_{i+1} \in E(G)$. Then we set $W_{i+1} = W_iy_ix_{i+1}$ and $W_{i+1}^* = W_iy_ix_{i+1}y_{i+1} = W_i^*x_{i+1}y_{i+1}$.

Finally, update $F_i = F \setminus (\{\text{ edges incident to } W_{i+1}\} \cup x_1y_{i+1})$.

We keep iterating until, after the final iteration I, the set F_I is empty. Since

$$F_I = F \setminus (\{\text{ edges incident to } W_I\} \cup x_1y_I)$$

and because the number of edges incident to W_I is at most $(\Delta - 1) \cdot |W_I| + 1$, it follows that $0 = |F_I| \geq |F| - 2 - (\Delta - 1) \cdot |W_I|$.

Because G is P_{k+3}−free, our constructed path W_I^* cannot be too large. More precisely, we must have $k+2 \geq |W_I^*| = |W_I| + 2$, and therefore

$$\omega(L(G)^2) = |F| \leq (\Delta - 1) \cdot |W_I| + 2 \leq (\Delta - 1) \cdot k + 2.$$

This concludes the proof for paths. As for cycles, we need to extend the argument slightly.

Suppose $\omega(L(G)^2) = |F| \geq l \cdot (\Delta - 1) + 3$. Then W_I^* is a path on $|W_I| + 2 \geq \frac{|F|-2}{\Delta-1} + 2 \geq l + 2 + \frac{1}{\Delta-1}$ vertices. Note that in the i−th iteration, the order of the path W_i^* is increased by either 1 or 2. Therefore there exists a $j \leq I$ such that $|W_j^*| \in \{l+2, l+3\}$.

From now on, let's call the edges of F *blue* and the edges in $E(G)\backslash F$ *red*. First we derive that it suffices to show the existence of a P_{l+3} that starts and ends on blue edges. Suppose G has a path A of order $l+3 \geq 6$ that starts with a blue edge a_1a_2 and ends on another blue edge $a_{l+2}a_{l+3}$. These (non-incident) blue edges must be within distance 2, so there must be an edge between them that isn't part of A. If $a_1a_{l+3} \in E(G)$, then $a_1a_2 \ldots a_{l+3}$ is a C_{l+3}. Similarly, if $a_1a_{l+2} \in E(G)$ or $a_2a_{l+3} \in E(G)$, then there is a C_{l+2}. Finally, if $a_2a_{l+2} \in E(G)$, then there is a C_{l+1}. So G contains a cycle of order $l+1, l+2$ or $l+3$, which is not allowed; contradiction.

So we may assume that $|W_j^*| = l+2$, and $|W_{j+1}^*| = l+4$. To finish the proof, we will derive that G then contains another path of order $l+3$, starting and ending on blue edges.

Write $W_j^* = w_1 \ldots w_{l+2}$. First, since $|W_{j+1}^*| - |W_j^*| = 2$, we must have that $W_{j+1}^* = W_j^* w_{l+3} w_{l+4}$, where $w_{l+2}w_{l+3}$ is a red edge and $w_{l+3}w_{l+4}$ is blue. Second, since w_1w_2 and $w_{l+3}w_{l+4}$ are at distance 2, there is an edge e^* between them. From this observation, we obtain the desired P_{l+3} unless $e^* = w_1w_{l+4}$. Third, w_1w_2 and $w_{l+1}w_{l+2}$ must be at distance 2 from eachother, so they are connected by an edge e^{**} that is not part of W_j^*. This yields a forbidden C_{l+2} or C_{l+1}, unless $e^{**} = w_2w_{l+1}$. Fourth, note that w_lw_{l+1} is red, for otherwise $w_{l+3}w_{l+4}w_1w_2 \ldots w_lw_{l+1}$ would yield the desired P_{l+3}.

In summary, we have obtained the cycle $\Gamma := w_{1+1} \ldots w_{l+4}$, where $w_1w_2, w_{l+1}w_{l+2}$ and $w_{l+3}w_{l+4}$ are blue, and w_lw_{l+1} is red. Furthermore, it holds that $w_2w_l \in E(G)$.

Next, we are going to focus on the edge $e^{***} := w_{l-1}w_l$. Since $l \geq 3$, this edge is different from the first edge w_1w_2. Suppose that e^{***} is blue.

Then $w_lw_{l-1} \ldots w_2w_{l+1}w_{l+2}w_{l+3}w_{l+4}$ forms a P_{l+3} starting and ending on blue edges. Suppose on the other hand that e^{***} is red. Because e^{***} and w_lw_{l+1} are consecutive red edges of W_{j+1}^*, it follows from the construction of the paths $(W_i^*)_{1 \leq i \leq j+1}$ that there must be a pending blue edge w_lw_p that is only incident to Γ in the vertex w_l. (This pending edge used to be the blue end-edge of some preliminary path W_i^*, $i < j$.) Now $w_pw_lw_{l-1} \ldots w_2w_1w_{l+4}w_{l+3}$ forms a P_{l+3}, starting and ending on blue edges.

□

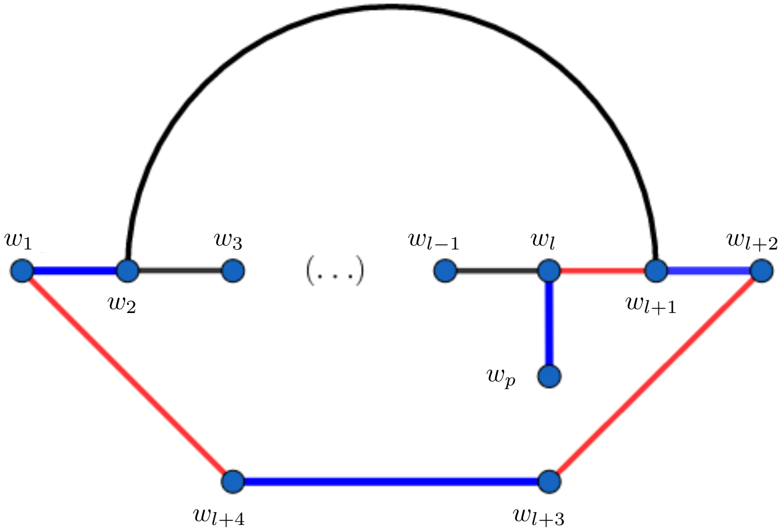

Figure 4.3: The subgraph constructed in the final part of the proof of Theorem 4.1.9, yielding a path of order $l+3$ starting and ending on blue edges, which in turn implies the existence of a C_{l+1}, C_{l+2} or C_{l+3}. The black edges are either red or blue

4.5 Bipartite graphs without an even cycle.

4.5.1 A large girth bipartite graph of large strong chromatic index

In this subsection we prove Lemma 4.1.10. The argument is taken almost literally from [82], with a few adaptations to ensure the obtained random graph is bipartite.

Proof of Lemma 4.1.10
Let $G = G(n,n,p)$ be a random bipartite graph on parts that are both of order n and with edge probability $p = d/n$, where d is sufficiently large and n is sufficiently large with respect to d. The expected number of cycles of length less than g in G is upperbounded by $gn^g p^g = gd^g$. So by Markov's inequality it follows that G has less than $\ln(n)$ cycles of length less than g, asymptotically almost surely. Second, the probability that G contains an induced matching of size $s := \frac{2n}{d}\ln(d)$ is at most

$$\begin{aligned}
\binom{n}{s}\frac{n!}{(n-s)!}\cdot p^s(1-p)^{s^2-s} &\leq \left(\frac{en}{s}\cdot np\cdot(1-p)^{s-1}\right)^s \\
&\leq \left(O(1)\cdot\frac{d^2}{\ln(d)}\cdot\left(1-\frac{d}{n}\right)^{2\frac{n}{d}\ln(d)-1}\right)^s \\
&\leq \left(O(1)\cdot\frac{d^2}{\ln(d)}\cdot e^{-2\ln(d)+1}\right)^s \leq \left(\frac{O(1)}{\ln(d)}\right)^s = o(1).
\end{aligned}$$

So G asymptotically almost surely does not contain an induced matching of size $\frac{2n}{d}\ln(d)$. Third, let A denote the number of edges in G adjacent to the vertices of degree at least $d+\frac{d}{\ln(d)}$. Then

$$\begin{aligned}\mathbb{E}(A) &\leq 2n\sum_{k=d+d/ln(d)}^{2n-1} k\binom{2n-1}{k}p^k(1-p)^{2n-1-k}\\ &= 2np(2n-1)\sum_{k=d+d/\ln(d)-1}^{2n-2}\binom{2n-2}{k}p^k(1-p)^{2n-2-k}.\end{aligned}$$

The right hand side equals $2np(2n-1)$ times the probability that a binomially distributed random variable with parameters $2n-2$ and p is at least $d-1+d/\ln(d)$. By a standard tail estimate (see eg [5], Appendix A), it follows that $\mathbb{E}(A)\leq(2n)^2p\cdot\frac{1}{d^2}=\frac{4n}{d}$. Applying Markov's inequality yields $\mathbb{P}(A\leq n)\geq 1-\frac{4}{d}$. Fourth and finally, note that the number of edges in G is $(1+o(1))nd$, asymptotically almost surely.

We now construct a new graph G_* by removing from G all the vertices contained in a cycle of order less than g or having degree at least $\Delta=d+d/\ln(d)$. We proved that, with positive probability, G_* has at least $(1+o(1))nd-n-(d+d/\ln(d))\cdot\ln(n)=(1+o(1))n\Delta$ edges, has girth at least g and contains no induced matching of size $\frac{2n}{d}\ln(d)=(2+o(1))(n/\Delta)\ln(\Delta)$. In particular, the stability number of $L(G_*)^2$ is at least $(2+o(1))(n/\Delta)\ln(\Delta)$. This implies that

$$\chi(L(G_*)^2)\geq\frac{|V(L(G_*)^2)|}{\alpha(L(G_*)^2)}\geq\frac{(1+o(1))n\Delta}{(2+o(1))(n/\Delta)\ln(\Delta)}=\left(\frac{1}{2}+o(1)\right)\frac{\Delta^2}{\ln(\Delta)}.$$

To finish the proof, note that by construction $\Delta\geq\Delta(G^*)$. □

4.5.2 Bipartite without C_4

Proof of Lemma 4.1.12.
Due to Theorem 4.1.3 the Lemma holds for $k\geq\Delta+1$, so we are left with the case $k=2$. Let G be a bipartite graph that is C_4-free or P_5-free (or both). We need to show that then $\omega(L(G)^2)\leq 2\Delta-1$, unless $\Delta=k=2$ and G is P_5-free but not C_4-free, in which case we need to show that $\omega(L(G)^2)\leq 2\Delta=4$.

Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$. Let $uv\in E(H)$. Note that $N(u)\cap N(v)=\emptyset$, for otherwise we have a triangle. Define

$$D(u):=\{x\in N_G(u)\setminus\{v\}\mid N_H(x)\setminus\{u\}\neq\emptyset\}$$

and analogously $D(v):=\{x\in N_G(v)\setminus\{u\}\mid N_H(x)\setminus\{v\}\neq\emptyset\}$.

First note that $|D(u)|\leq 1$ (and symmetrically, $|D(v)|\leq 1$). Indeed, suppose for a contradiction that $|D(u)|\geq 2$. Then there are $x_1,x_2\in N_G(u)\setminus\{v\}$ with neighbours $y_1\in N_H(x_1)\setminus\{u\},y_2\in N_H(x_2)\setminus\{u\}$. Since $x_1y_1,x_2y_2\in E(H)$ they are either incident (in which case $y_1=y_2$ so that $ax_1y_1x_2$ is a C_4 and $vux_1y_1x_2$ is a P_5) or at distance 2 (in which case we may assume by symmetry that $x_1y_2\in E(G)$, so that $ux_1y_1y_2$ is a C_4 and $y_1x_1ux_2y_2$ is a P_5). Contradiction.

Second,

$$\text{if } D(u) \neq \emptyset, \text{ then } N_H(v)\setminus\{u\} = \emptyset \tag{4.6}$$

(and symmetrically, if $D(v) \neq \emptyset$, then $N_H(u)\setminus\{v\} = \emptyset$). Indeed, suppose for a contradiction that $D(u) \neq \emptyset$ and $N_H(v)\setminus\{u\} \neq \emptyset$. Then there exist $x \in N_G(u)$, $y \in N_H(x)\setminus\{u,v\}$ and $z \in N_H(v)\setminus\{u\}$. If $y = z$ then $uvxz$ is a C_4 and any extra edge creates a P_5. So $|E(H)| \leq 4$ and this bound is only attained if $\Delta = 2$ (this is the only place where the exceptional case $k = \Delta$ manifests itself). Thus, we may assume that $y \neq z$. Since the edges xy, bz are in $E(H)$ and not incident, there is an edge between them. Since the graph is bipartite, this is not possible without creating a C_4 and a P_5. Contradiction.

Now we have all the ingredients to finish the proof with a simple case analysis. If $D(u), D(v) \neq \emptyset$, then by (4.6), uv is the only edge of H incident to $\{u,v\}$. So $|E(H)| \leq |\{uv\}| + |N_H(D(u)\setminus\{u\}| + |N_H(D(v)\setminus\{v\}| \leq 1 + 2\cdot(\Delta - 1) = 2\Delta - 1$. If $D(u) \neq \emptyset, D(v) = \emptyset$, then $|E(H)| \leq |N_H(u)| + |N_H(D(u))\setminus\{u\}| \leq \Delta + \Delta - 1 = 2\Delta - 1$. Finally, if $D(u) = D(v) = \emptyset$, then all edges of H are incident to uv, so again $|E(H)| \leq 2\Delta - 1$.

□

4.5.3 Bipartite without even cycle

Proof of Theorem 4.1.13
By Theorem 4.1.3, we may assume throughout that $k \leq \Delta$. Let $G = G[X,Y]$ be bipartite and P_{2k+1}−free. Let H be a subgraph of G whose edges form a maximum clique in $L(G)^2$, so that $|E(H)| = \omega(L(G)^2$. A path in G will be called *H-sided* if it starts and ends on edges of H. Given a vertex $v \in V(G)$, an *H−neighbour* of v is a vertex $w \in N_H(v)$.

Assume that $\omega(L(G)^2) > \max(k\Delta, 2k(k-1))$. Under this assumption, we want to derive that for any H−sided path P of order smaller than $2k+1$, we can find another H−sided path that has order $|P|+1$ or $|P|+2$, which is sufficient by the following claim.

Claim 4.5.1. *Suppose that for each H−sided path P in G of order $|P| < 2k+1$, we can find another H−sided path of order $|P|+1$ or $|P|+2$. Then G contains P_{2k+1} as a subgraph, and also contains a copy of C_{2k+2} or C_{2k}.*

Proof. Because $|E(H)| \geq 1$, there exists an H−sided path of order 2. We can iteratively extend the length of this path by 1 or 2, ultimately yielding an H−sided path P of order $\in \{2k+1, 2k+2\}$. In particular, G contains a path of order $2k+1$, as desired. The first and final edge of P are in H and therefore (also using that $|P| \geq 2k+1 \geq 5$) they must be at distance *exactly* 2. Since G is bipartite, this implies the existence of a cycle of order $\in \{|P|, |P|-2\}$ if $|P|$ is even, and a cycle of order $|P|-1$ if $|P|$ is odd. So G has a cycle of order $\in \{2k, 2k+2\}$. □

Let P be an H−sided path. For clarity of notation we assume from now on that P has even order $2l$, for some $l \leq k$. For paths of odd order $< 2k+1$ the arguments are similar and in fact slightly easier, because the bounds we need are slightly more forgiving in that case. Write $P := p_1p_2 \dots p_{2l}$.

First, we need to introduce some definitions. Let $X_P := X \cap V(P) = p_1p_3 \dots p_{2l-1}$ and $Y_P := Y \cap V(P) = p_2p_4 \dots p_{2l}$ be the two parts of the bipartite graph induced by P. A vertex of P will be called *$r-$extravert* if its number of H-neighbours outside P is at least r. For short, we call the vertex *extravert* if it is 1-extravert. Conversely, a vertex of P is *introvert* if all of its H-neighbours are in P. By $P_{\text{ext}}^{(r)}$ and P_{ext} we denote the set of $r-$extravert vertices and extravert vertices respectively, and P_{int} denotes the set of introvert vertices. Finally, by $\text{Obs}(P)$ we will denote the set of *obsolete edges*, which by definition are those edges of H that are incident to some vertex of $P \setminus \{p_1, p_{2l}\}$. We call them obsolete because they cannot be 'greedily' used to extend the order of P.

From now on, suppose for a contradiction that it is *not* possible to find an $H-$sided path of order $|P|+1$ or $|P|+2$. Then the following claims hold.

Claim 4.5.2. *The first and final vertex of P are introvert.*

Proof. Suppose by symmetry that the first vertex p_1 is extravert. Then it has an $H-$neighbour p_0 outside P, so p_0P is an $H-$sided path of order $|P|+1$. Contradiction. □

Claim 4.5.3. $|Obs(P)| > \max(k\Delta, 2k(k-1))$.

Proof. Suppose not. Then $|\text{Obs}(P)| \leq \max(k\Delta, 2k(k-1)) < |E(H)|$. Therefore there exists an edge e^* in H that is not incident to any vertex of P. The final edge e of P is in H, so e^* and e must be at distance *exactly* 2. This implies that we can extend P to an $H-$sided path (ending on e^* rather than e) that is of order $|P|+1$ or $|P|+2$. Contradiction. □

So in order to arrive at a contradiction, it suffices to show that $|\text{Obs}(P)| \leq k\Delta$ or $|\text{Obs}(P)| \leq 2k(k-1)$. We will now derive some structural properties of our counterexample.

Claim 4.5.4. *Any two extravert vertices in the same part (both in X_P or both in Y_P) have a common neighbour outside P.*

Proof. Indeed, suppose wlog that p_i, p_j are two extravert vertices in X_P, with $H-$neighbours q_i respectively q_j outside P. If $q_i = q_j$ we are done, so suppose $q_i \neq q_j$. The edges p_iq_i and p_jq_j need to be within distance 2. Since odd cycles are not allowed in G, it follows that $p_ip_j, q_iq_j \notin E(G)$, so q_i or q_j must be a common neighbour of p_i and p_j. □

Claim 4.5.5. *P contains at most two pairs of consecutive extravert vertices, and if there are two such pairs p_ip_{i+1} and p_jp_{j+1}, then they must have different parity, in the sense that $i = j+1$ (mod 2).*

Proof. Suppose there are two extravert pairs p_ip_{i+1}, p_jp_{j+1} of the same parity. Then wlog $i+1 < j$ and $p_i, p_j \in X_P$ and $p_{i+1}, p_{j+1} \in Y_P$. See 4.4. By Claim 4.5.4, p_i and p_j have a common neighbour $u \in Y \setminus Y_P$, and p_{i+1} and p_{j+1} have a common neighbour $v \in X \setminus X_P$. Therefore we can replace the subpath $P^* := p_ip_{i+1}, \dots p_jp_{j+1}$ of P by $p_iup_jp_{j-1} \dots p_{i+2}p_{i+1}vp_{j+1}$, which uses the same vertices as P^* *and* two extra vertices u, v outside of P. Thus, we have constructed an $H-$sided path of order $|P|+2$. Contradiction. □

The next claim is arguably the heart of the argument.

Claim 4.5.6. *There are at most l extravert vertices.*

Proof. Consider the vertex pairs $(p_2, p_3), (p_4, p_5), \ldots (p_{2l-2}, p_{2l-1})$. By Claim 4.5.2, all extravert vertices are contained in the union of these $l-1$ pairs. So if there are more than l extravert vertices, then by the pigeonhole principle at least two pairs entirely consist of extravert vertices. We have obtained two same-parity pairs of consecutive extravert vertices, contradicting Claim 4.5.5.

□

From now on, let $r \in \mathbb{N}_{\geq 0}$ be the maximal integer (if it exists) such that there are nonadjacent $r-$extravert vertices s, t with $s \in X_P$ and $t \in Y_P$.

Claim 4.5.7. *The integer r is well-defined.*

Proof. Suppose r does not exist. Then the vertices of P induce a complete bipartite graph, with parts X_P and Y_P. By Claim 4.5.6 we have $|P_{\text{ext}}| \leq l$, and therefore $|\text{Obs}(P)| \leq |X_P| \cdot |Y_P| + |P_{\text{ext}}| \cdot (\Delta - \min(|X_P|, |Y_P|)) = l^2 + l \cdot (\Delta - l) \leq k \cdot \Delta$, contradicting Claim 4.5.3. □

The next claim follows directly from the definition of r.

Claim 4.5.8. *The graph induced by $P_{ext}^{(r+1)}$ is complete bipartite.*

For vertices a, b in P, let $d_P(a, b)$ denote the number of edges in the subpath of P having endpoints a and b. We will call $d_P(a, b)$ the *P-distance* between a and b.

Claim 4.5.9. *Let $q \in \mathbb{N}_{\geq 1}$. Let $a \in X_P, b \in Y_P$ be two non-adjacent $q-$extravert vertices. Then they are at P-distance at least $2q+1$.*

Proof. Suppose for a contradiction that $d := d_P(a, b) + 1 \leq 2q$. Note that d is the (even) number of vertices in the subpath of P between (and including) a and b. Let $A := \{a_1, \ldots, a_q\}$ denote a subset of the $H-$neighbours of a in $Y \backslash Y_P$. Similarly, let $B := \{b_1, \ldots, b_q\}$ denote a subset of the $H-$neighbours of b in $X \backslash X_P$. See figure 4.4. Because $ab \notin E(G)$ and the $H-$edges $a_i a, b_j b$ should be within distance 2 for all i, j, it follows that A is complete to B. Therefore there exists a path $P^* := a a_1 b_1 a_2 b_2 \ldots a_q b_q b$ of order $d+2$ that only intersects P in a and b. This leads to a contradiction, because it implies that we can construct an $H-$sided path of order $|P|+2$, by replacing the order d subpath of P between a and b with the order $d+2$ path P^*. □

With the above claims, we will now complete the proof of Theorem 4.1.13 by deriving a contradiction to Claim 4.5.3.

We will partition the vertices of P and estimate the $H-$edges incident to them separately. First we need some definitions. Let $i_x := |P_{\text{ext}}^{(r+1)} \cap X_P|$ and $i_y := |P_{\text{ext}}^{(r+1)} \cap Y_P|$ be the numbers of $(r+1)-$extravert vertices in the parts X_P, Y_P of the bipartite graph induced by P. Similarly, let $j_x := |P_{\text{ext}} \backslash P_{\text{ext}}^{(r+1)} \cap X_P|$ and $j_y := |P_{\text{ext}} \backslash P_{\text{ext}}^{(r+1)} \cap Y_P|$ be the number of vertices that are extravert but not $(r+1)-$extravert, in part X_P respectively Y_P. Note that the remaining $|X_P| - i_x - j_x$ (resp. $|Y_P| - i_y - j_y$) vertices in X_P (resp. Y_P) are introvert.

An important observation is that we can write $\mathrm{Obs}(P)$ as a disjoint union $E_1 \cup E_2 \cup E_3$, where

$$E_1 = \left\{ H\text{-edges incident to } P_{\text{ext}}^{(r+1)} \right\},$$

$$E_2 = \left\{ H\text{-edges incident to } P_{\text{ext}} \text{ but not incident to } P_{\text{ext}}^{(r+1)} \right\} \text{ and}$$

$$E_3 = \left\{ H\text{-edges in the graph induced by } P_{\text{int}} \right\}.$$

See figure 4.5.

Recall from Claim 4.5.8 that $G[P_{\text{ext}}^{(r+1)}]$ is complete bipartite, so it is efficient to estimate E_1 by summing the degrees (with respect to G) of $P_{\text{ext}}^{(r+1)}$ and subtracting the double-counted edges of $G[P_{\text{ext}}^{(r+1)}]$. This yields

$$|E_1| \leq -\left| E_G[P_{\text{ext}}^{(r+1)}] \right| + \sum_{v \in P_{\text{ext}}^{(r+1)}} |N_G(v)| \leq -i_x i_y + (i_x + i_y) \cdot \Delta. \tag{4.7}$$

To estimate $|E_2|$, note that it is maximized if each vertex $v \in P_{\text{ext}} \backslash P_{\text{ext}}^{(r+1)}$ has exactly r $H-$neighbours outside $G[P]$ and is incident to all vertices of the opposite part that are not in $P_{\text{ext}}^{(r+1)}$ (and leaving out one single edge from this graph, to comply with the non-edge that defines r). In this case

$$\begin{aligned} |E_2| &\leq -\left| E_H[P_{\text{ext}} \backslash P_{\text{ext}}^{(r+1)}] \right| + \sum_{v \in P_{\text{ext}} \backslash P_{\text{ext}}^{(r+1)}} \left| N_H(v) \backslash P_{\text{ext}}^{(r+1)} \right| \qquad (4.8) \\ &\leq -j_x j_y + j_x \cdot (r + |Y_P| - i_y) + j_y \cdot (r + |X_P| - i_x). \end{aligned}$$

The size of E_3 is maximized if P_{int} induces a complete bipartite graph, so

$$|E_3| \leq (|X_P| - i_x - j_x) \cdot (|Y_P| - i_y - j_y). \tag{4.9}$$

Summing estimates (4.7), 4.8) and (4.9), we conclude that

$$\begin{aligned} |\mathrm{Obs}(P)| &= |E_1| + |E_2| + |E_3| \\ &\leq (i_x + i_y) \cdot \Delta + (j_x + j_y) \cdot r + |X_P| \cdot |Y_P| - i_x \cdot |Y_P| - i_y \cdot |X_P| \\ &= (i_x + i_y) \cdot (\Delta - l) + (j_x + j_y) \cdot r + l^2. \qquad (4.10) \end{aligned}$$

If $\Delta - l \geq r$ then (4.10) is maximized for $j_x + j_y = 0$, so that $i_x + i_y = |P_{\text{ext}}|$. This means that all extravert vertices are in fact $(r+1)-$extravert. By Claim 4.5.6,

$$|\mathrm{Obs}(P)| \leq |P_{\text{ext}}| \cdot (\Delta - l) + l^2 \leq l \cdot (\Delta - l) + l^2 \leq k \cdot \Delta,$$

a contradiction to Claim 4.5.3. Conversely, if $\Delta - l < r$ then the upperbound on $|\mathrm{Obs}(P)|$ is maximized for $i_x + i_y = 0$, so that $j_x + j_y = |P_{\text{ext}}|$. This means that none of the extravert vertices is $(r+1)-$extravert. By Claim 4.5.6, we again obtain a contradiction to Claim 4.5.3:

$$|\mathrm{Obs}(P)| \leq |P_{\text{ext}}| \cdot r + l^2 \leq l \cdot (l-2) + l^2 \leq 2k(k-1). \tag{4.11}$$

In the last line, we used that $r \leq l - 2$, which follows from Claim 4.5.9 and the fact that the first and final vertex of P are introvert.

□

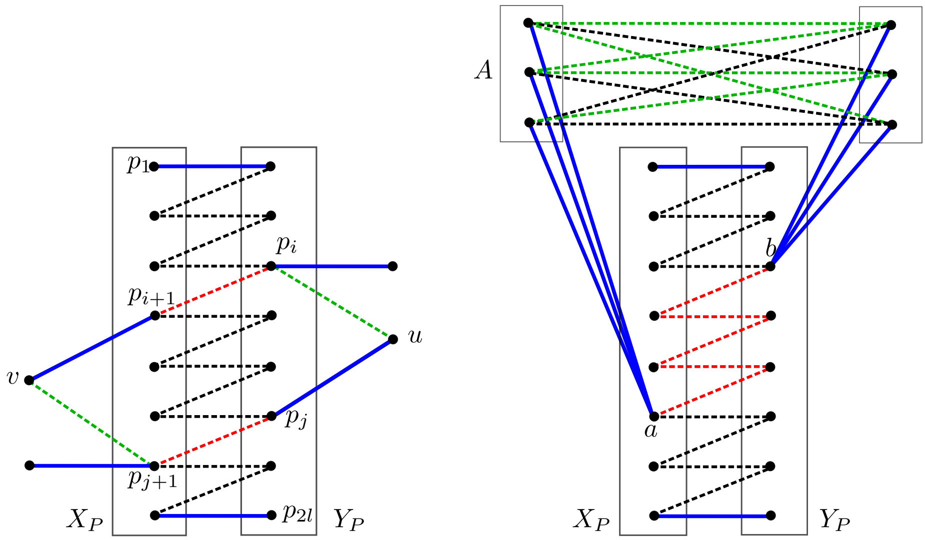

Figure 4.4: A depiction of the contradictory path-extensions described by Claim 4.5.5 (left) and Claim 4.5.9 (right). On the right, a and b are non-adjacent 3−extravert vertices and $d_P(a, b) = 5$. This means that a and b are too close to eachother. Indeed, by following the green edges (and two blue edges) rather than the red edges, we obtain an H−sided path of order $|P| + 2$.

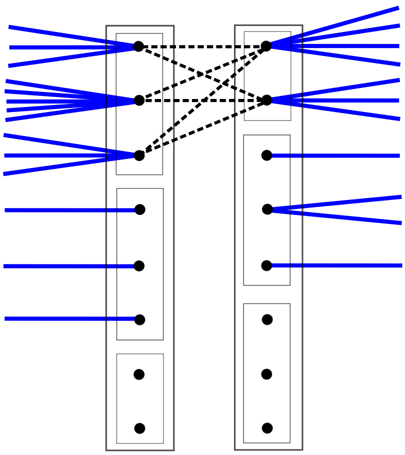

Figure 4.5: A simplified depiction of the structure described on page 78, in case $r = 2$. From top to bottom we have the 3−extravert vertices, the extravert vertices that are not 3−extravert, and finally the introvert vertices. The three sets on the left are the sets counted by i_x, j_x and $|X_P| - i_x - j_x$. The three sets on the right are counted by j_x, j_y and $|Y_P| - i_y - j_y$. The union of the two upper sets equals $P_{\text{ext}}^{(r+1)}$ and thus induces a *complete* bipartite graph.

4.6 Bounded Hadwiger number

4.6.1 Graphs with bounded Hadwiger number

In this section we prove Theorem 4.1.16.

Proof of Theorem 4.1.16.
Let H be a subgraph of G whose edges represent a maximum clique in $L(G)^2$, so that $\omega(L(G)^2) = |E(H)|$. Let $C \subseteq E(H)$ be a maximum set of edges that are pairwise at distance precisely two. By contracting each edge in C, we obtain a clique of order $|C|$, so G has $K_{|C|}$ as a minor. Therefore $|C| \leq k - 1$. By the maximality of C, every edge of H must be incident to an edge of C. Note that from this we already obtain $\omega(L(G)^2) = |E(H)| \leq |C| \cdot (2\Delta - 1) \leq (k-1)(2\Delta - 1)$. We need to improve on this bound by roughly a factor two, under the additional assumption that $\Delta \geq 2k - 1$.

Let $V(C)$ denote the set of vertices incident to an edge of C. A vertex $a \in V(C)$ is called $k-$*extravert* if $|N_H(a) \backslash V(C)| = k$ and *extravert* for short if it is $k-$extravert for some $k \geq 1$. [1] Consider an edge $ab \in C$, where a is $k-$extravert and b is $l-$extravert for some $0 \leq k \leq l$. We call a the *shy* vertex of ab and b the *outgoing* vertex of ab (and if $k = l$ we assign an arbitrary vertex of ab to be the shy vertex). Furthermore, we call the edge ab of *type* 1 if it satisfies one of the following.

- Both a and b are $1-$extravert with a common neighbour in $(N_H(a) \cap N_H(b)) \backslash V(C)$.

- a is not extravert and b is $(\leq 1)-$extravert.

We call ab of *type* 2 if a is not extravert and b is $k-$extravert for some $k \geq 2$. Correspondingly, we call a vertex of type 1 (respectively 2) if it is incident to an edge of type 1 (respectively 2).

First, we observe that all edges of C are either of type 1 or 2. Indeed, if not, then there is an edge $ab \in C$ such that a is $(\geq 1)-$extravert and b is $(\geq 2)-$extravert. Therefore there exist two distinct vertices $p, q \notin V(C)$ such that $ap, bq \in E(H)$. But both ap and bq are at distance exactly two from eachother and from the edges in $C \backslash \{ab\}$, so $C \cup \{ap, bq\} \setminus \{ab\}$ contradicts the maximality of C.

Second, we note that whenever there is an edge $a_1a_2 \in E(H)$ of H between shy vertices a_1, a_2, it must hold that both shy vertices are of type 1. Indeed, let a_1b_1, a_2b_2 be the corresponding edges of C and suppose for a contradiction that at least one of a_1b_1, a_2b_2 is of type 2. Then there exist *distinct* vertices $z_1, z_2 \notin V(C)$ such that $z_1b_1, z_2b_2 \in E(H)$. Now $C \cup \{z_1b_1, z_2b_2, a_1a_2\} \setminus \{a_1b_1, a_2b_2\}$ contradicts the maximality of C.[2]

[1] Note that this definition of a $k-$*extravert* vertex differs slightly from the definition used in section 4.5.3 !

[2] Note that this argument fails to forbid edges of H between two shy vertices of type 1, because in that case we cannot guarantee that $z_1 \neq z_2$, so that z_1b_1 and z_2b_2 may be incident. As a consequence, we may not be allowed to add both z_1b_1 and z_2b_2 to C.

By the properties derived above, it follows that we can partition $E(H) = E_1 \cup E_2 \cup E_3$, where

$$E_1 := \{e \in E(H) \mid e \text{ incident to an outgoing vertex of type } 1\},$$

$$E_2 := \{e \in E(H) \mid e \text{ incident to an outgoing vertex of type } 2\} \text{ and}$$

$$E_3 := \{a_1a_2 \in E(H) \mid a_1 \text{ is shy of type 1 and either } a_2 \notin V(C) \text{ or } a_2 \text{ is shy of type } 1\}.$$

We upper bound each of these edge sets separately. Let t_i denote the number of edges of type i and note that $t_1+t_2 = |C|$. Clearly we have $|E_2| \leq t_2 \cdot \Delta$. Since each outgoing vertex of type 1 is incident to at most $2|C|$ edges of H, of which at most one is not induced by $V(C)$, it follows that $|E_1| \leq t_1 \cdot 2|C| - \binom{t_1}{2}$. Finally, the shy vertices of type 1 induce at most $\binom{t_1}{2}$ edges, and each shy vertex of type 1 is $(\leq 1)-$extravert, so $|E_3| \leq \binom{t_1}{2} + t_1$. In conclusion, we obtain

$$\begin{aligned} |E(H)| \leq |E_1| + |E_2| + |E_3| &\leq t_1 \cdot 2|C| - \binom{t_1}{2} + t_2 \cdot \Delta + \binom{t_1}{2} + t_1 \\ &= t_1 \cdot (2|C| + 1 - \Delta) + |C| \cdot \Delta \\ &\leq |C| \cdot \max(\Delta, 2|C| + 1). \end{aligned}$$

Since $|C| \leq k - 1$ and (by assumption) $\Delta \geq 2k - 1$, it follows that $\omega(L(G)^2) = |E(H)| \leq |C| \cdot \Delta \leq (k-1) \cdot \Delta$.

□

Remark. Albeit close to sharp, there is still some room to improve on the upper bound given in Theorem 4.1.16, since e.g. we disregard some overcounting that could arise from the edges induced by the outgoing vertices of type 2. Also, we did not use the fact that if there are many edges in the graph induced by $V(C)$, then K_t is a minor for some t strictly larger than $|C|$.

4.6.2 Multigraphs with bounded Hadwiger number

In this section we prove Lemma 4.1.18. Given a multigraph G, let $\mu(G)$ denote the matching number of G (i.e. the size of the largest matching) and let $o(G)$ denote the number of components of G that have an odd number of vertices. According to the Tutte-Berge formula [10], it holds for any multigraph $G = (V, E)$ that $\mu(G) = \frac{1}{2} \min_{U \subseteq V} (|U| - o(G - U) + |V|)$. Using this, we can derive Lemma 4.1.18.

Proof of Lemma 4.1.18.
First, note that for every vertex $v \in G$, the graph $G - v$ has at most one component with an edge. For if there are two components with an edge, then these edges cannot be within distance two in G. More generally, for any $U \subseteq V(G)$, the graph $G - U$ has at most one component with an edge.

Second, note that $\mu(G) \leq h(G)$. Indeed, suppose for a contradiction that G has a matching M of size $|M| > h(G)$. Since there must be an edge between any two edges $e_1, e_2 \in M$, contracting each edge of M to a vertex yields a complete graph on $|M|$ vertices. Therefore the complete graph on $h(G)+1$ vertices is a minor of G; contradiction.

Let $U \subseteq V(G)$ be a vertex subset such that $\frac{1}{2}(|U| - o(G-U) + |V|) = \mu(G)$. If $G-U$ has no edges, then $o(G-U) = |V| - |U|$, so that $\mu(G) = |U|$ and $|E(G)| \leq \sum_{u \in U} \deg(u) \leq \Delta(G) \cdot |U| \leq \Delta(G) \cdot h(G)$. So we may assume $G-U$ has a component with an edge. Let A denote the set of vertices in this (unique) edged component. If $|A|$ is even then

$$\mu(G) = \frac{1}{2}(|U| - o(G-U) + (|U| + o(G-U) + |A|)) = |U| + \frac{|A|}{2}.$$

On the other hand, if $|A|$ is odd then

$$\mu(G) = \frac{1}{2}(|U| - o(G-U) + (|U| + o(G-U) + |A| - 1)) = |U| + \frac{|A|}{2} - \frac{1}{2}.$$

It follows that

$$\begin{aligned} |E(G)| &\leq |E[A]| + \sum_{u \in U} \deg(u) \leq \Delta(G) \cdot \left(|U| + \frac{|A|}{2}\right) \leq \Delta(G) \cdot \left(\mu(G) + \frac{1}{2}\right) \\ &\leq \Delta(G) \cdot \left(h(G) + \frac{1}{2}\right). \end{aligned}$$

□

Chapter 5

Colouring Jordan regions and curves

A Jordan region is a subset of the plane that is homeomorphic to a closed disk. Consider a family $\mathcal{F}$ of Jordan regions whose interiors are pairwise disjoint, and such that any two Jordan regions intersect in at most one point. If any point of the plane is contained in at most k elements of $\mathcal{F}$ (with k sufficiently large), then we show that the elements of $\mathcal{F}$ can be coloured with at most $k+1$ colours so that intersecting Jordan regions are assigned distinct colours. This is best possible and answers a question raised by Reed and Shepherd in 1996. As a simple corollary, we also obtain a positive answer to a problem of Hliněný (1998) on the chromatic number of contact systems of strings. We also investigate the chromatic number of families of touching Jordan curves. This can be used to bound the ratio between the maximum number of vertex-disjoint directed cycles in a planar digraph, and its fractional counterpart.

5.1 Introduction

In this chapter, a *Jordan region* is a subset of the plane that is homeomorphic to a closed disk. A family $\mathcal{F}$ of Jordan regions is *touching* if their interiors are pairwise disjoint. If any point of the plane is contained in at most k Jordan regions of $\mathcal{F}$, then we say that $\mathcal{F}$ is *k-touching*. If any two elements of $\mathcal{F}$ intersect in at most one point, then $\mathcal{F}$ is said to be *simple*. All the families of Jordan regions and curves we consider are assumed to have a finite number of intersection points. The first part of this chapter is concerned with the *chromatic number* of simple k-touching families of Jordan regions, i.e. the minimum number of colours needed to colour the Jordan regions, so that intersecting Jordan regions receive different colours. This can also be defined as the chromatic number of the *intersection graph* $G(\mathcal{F})$ of $\mathcal{F}$, which is the graph with vertex set $\mathcal{F}$ in which two vertices are adjacent if and only if the corresponding elements of $\mathcal{F}$ intersect. Recall that the *chromatic number* of a graph G, denoted by $\chi(G)$, is the least number of colours needed to colour the vertices of G, so that adjacent vertices receive different colours. The chromatic number of a graph G is at least the *clique number* of

G, denoted by $\omega(G)$, which is the maximum number of pairwise adjacent vertices in G, but the difference between the two parameters can be arbitrarily large (see [75] for a survey on the chromatic and clique numbers of geometric intersection graphs).

The following question was raised by Reed and Shepherd [92].

Problem 5.1.1. *[92] Is there a constant C such that for any simple touching family $\mathcal{F}$ of Jordan regions, $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + C$? Can we take $C = 1$?*

Our main result is the following (we made no real effort to optimize the constant 490, which is certainly far from optimal, our main concern was to give a proof that is as simple as possible).

Theorem 5.1.2. *For $k \geq 490$, any simple k-touching family of Jordan regions is $(k+1)$-colourable.*

Note that apart from the constant 490, Theorem 5.1.2 is best possible. Figure 5.1 depicts two examples of simple k-touching families of Jordan regions of chromatic number $k+1$.

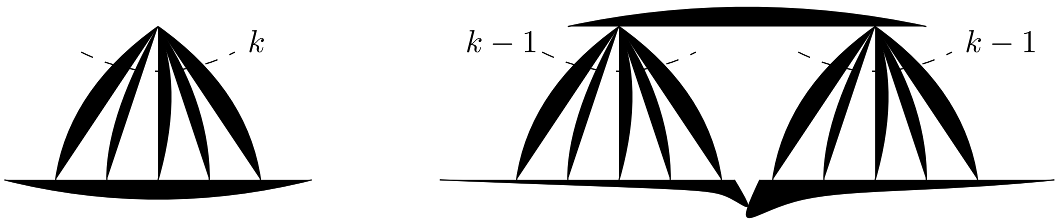

Figure 5.1: Two simple k-touching families of Jordan regions with chromatic number $k+1$.

It was proved in [41] that every simple k-touching family of Jordan regions is $3k$-colourable (their result is actually stated for k-touching families of strings, but it easily implies the result on Jordan regions). We obtain the next result as a simple consequence.

Corollary 5.1.3. *Any simple k-touching family of Jordan regions is $(k+327)$-colourable.*

Proof. Let $\mathcal{F}$ be a simple k-touching family $\mathcal{F}$ of Jordan regions. If $k \leq 163$ then $\mathcal{F}$ can be coloured with at most $3k \leq k+327$ colours by the result of [41] mentioned above. If $164 \leq k \leq 489$, then $\mathcal{F}$ is also 490-touching, and it follows from Theorem 5.1.2 that $\mathcal{F}$ can be coloured with at most $491 \leq k+327$ colours. Finally, if $k \geq 490$, Theorem 5.1.2 implies that $\mathcal{F}$ can be coloured with at most $k+1 \leq k+327$ colours. $\square$

Observe that for a given simple touching family $\mathcal{F}$ of Jordan regions, if we denote by k the least integer so that $\mathcal{F}$ is k-touching, then $\omega(G(\mathcal{F})) \geq k$, since k Jordan regions intersecting some point p of the plane are pairwise intersecting. Therefore, we obtain the following immediate corollary, which is a positive answer to the problem raised by Reed and Shepherd.

Corollary 5.1.4. *For any simple touching family $\mathcal{F}$ of Jordan regions, $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 327$ (and $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 1$ if $\omega(G(\mathcal{F})) \geq 490$).*

Note that the bound $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 1$ is also best possible (as shown by Figure 5.1, right).

It turns out that our main result also implies a positive answer to a question raised by Hliněný in 1998 [60]. A *string* is the image of some continuous injective function from $[0, 1]$ to $\mathbb{R}^2$, and the *interior* of a string is the string minus its two endpoints. A *contact systems of strings* is a set of strings such that the interiors of any two strings have empty intersection. In other words, if c is a contact point in the interior of a string s, all the strings containing c distinct from s end at c. A contact system of strings is said to be *one-sided* if for any contact point c as above, all the strings ending at c leave from the same side of s (see Figure 5.2, left). Hliněný [60] raised the following problem:

Problem 5.1.5. *[60] Let $\mathcal{S}$ be a one-sided contact system of strings, such that any point of the plane is in at most k strings, and any two strings intersect in at most one point. Is it true that $G(\mathcal{S})$ has chromatic number at most $k + o(k)$? (or even $k + c$, for some constant c?)*

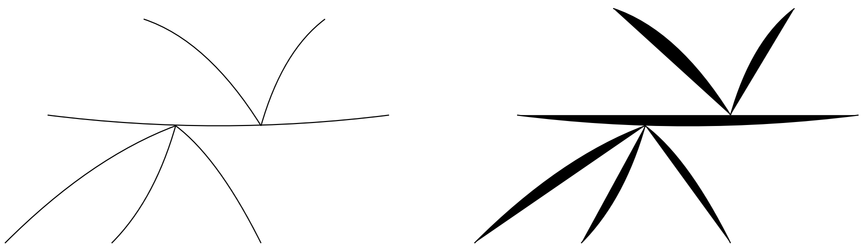

Figure 5.2: Turning a one-sided contact system of strings into a simple touching family of Jordan regions.

The following simple corollary of Theorem 5.1.2 gives a positive answer to Problem 5.1.5.

Corollary 5.1.6. *Let $\mathcal{S}$ be a one-sided contact system of strings, such that any point of the plane is in at most k strings, and any two strings intersect in at most one point. Then $G(\mathcal{S})$ has chromatic number at most $k + 127$ (and at most $k + 1$ if $k \geq 490$).*

Proof. Assume first that $k \leq 363$. It was proved in [41] that $G(\mathcal{S})$ has chromatic number at most $\left\lceil \frac{4}{3}k \right\rceil + 6$, so in this case at most $k + 127$, as desired. Assume now that $k \geq 364$. Let $\mathcal{F}$ be obtained from $\mathcal{S}$ by thickening each string s of $\mathcal{S}$, turning s into a (very thin) Jordan region (see Figure 5.2, from left to right). Since $\mathcal{S}$ is one-sided, each intersection point contains precisely the same elements in $\mathcal{S}$ and $\mathcal{F}$, and therefore $G(\mathcal{S})$ and $G(\mathcal{F})$ are equal, while $\mathcal{F}$ is a simple k-touching family of Jordan regions. If $364 \leq k \leq 489$, then $\mathcal{F}$ is also 490-touching and it follows from Theorem 5.1.2 that $G(\mathcal{S}) = G(\mathcal{F})$ has chromatic number at most $491 \leq k + 127$. Finally, if $k \geq 490$, then by Theorem 5.1.2, $G(\mathcal{S}) = G(\mathcal{F})$ has chromatic number at most $k + 1$, as desired. □

A *Jordan curve* is the boundary of some Jordan region of the plane. We say that a family of Jordan curves is *touching* if for any two Jordan curves a, b, the curves a and b do not cross (equivalently, either the interiors of the regions bounded by a and b are disjoint, or one is contained in the other). Moreover, if any point of the plane is on at most k Jordan curves, we say that the family is k-touching. Note that unlike above, the families of Jordan curves we consider here are not required to be simple (two Jordan curves may intersect in several points). Note that previous works on intersection of Jordan curves have usually considered the opposite case, where every two curves that intersect also cross (see for instance [66] and the references therein).

Let $\mathcal{F}$ be a k-touching family of Jordan curves. For any two intersecting Jordan curves $a, b \in \mathcal{F}$, let $\mathcal{D}(a, b)$ be the set of Jordan curves c distinct from a, b such that the (closed) region bounded by c contains exactly one of a, b. The cardinality of $\mathcal{D}(a, b)$ is called *the distance between* a and b, and is denoted by $d(a, b)$. Note that since $\mathcal{F}$ is k-touching, any two intersecting Jordan curves are at distance at most $k - 2$.

Given a k-touching family $\mathcal{F}$, the *average distance in* $\mathcal{F}$ is the average of $d(a, b)$, over all pairs of intersecting Jordan curves $a, b \in \mathcal{F}$. We conjecture the following.

Conjecture 5.1.7. *For any k-touching family $\mathcal{F}$ of Jordan curves, the average distance in $\mathcal{F}$ is at most $\frac{k}{2}$.*

It was proved by Fox and Pach [46] that each k-touching family of strings is $(6ek+1)$-colourable, which directly implies that each k-touching family of Jordan curves is $(6ek + 1)$-colourable (note that $6e \approx 16.31$). We show how to improve this bound when the average distance is at most αk, for some $\alpha \leq 1$.

Theorem 5.1.8. *Let $\mathcal{F}$ be a k-touching family of Jordan curves, such that the average distance in $\mathcal{F}$ is at most αk, for some constant $0 \leq \alpha \leq 1$. Then the chromatic number of $\mathcal{F}$ is at most $\frac{6e^\delta}{\delta+\delta^2(1-\alpha)}\, k$, where $\delta = \delta(\alpha) = \frac{1}{2-2\alpha}(1-2\alpha+\sqrt{4\alpha^2 - 8\alpha + 5})$ for $\alpha < 1$ and $\delta(1) = 1$.*

Note that $\delta(1) = 1 = \lim_{\alpha\to 1} \frac{1}{2-2\alpha}(1 - 2\alpha + \sqrt{4\alpha^2 - 8\alpha + 5})$. Theorem 5.1.8 has the following direct corollary.

Corollary 5.1.9. *Let $\mathcal{F}$ be a k-touching family of Jordan curves, such that the average distance in $\mathcal{F}$ is at most αk. Then $\mathcal{F}$ is $\beta\, k$-colourable, where*

$$\beta = \begin{cases} 12.76 & \text{if} \quad \alpha \leq 3/4, \\ 10.22 & \text{if} \quad \alpha \leq 1/2, \\ 8.43 & \text{if} \quad \alpha \leq 1/4. \end{cases}$$

By Corollary 5.1.9, a direct consequence of Conjecture 5.1.7 would be that every k-touching family of Jordan curves is $10.22\,k$-colourable.

For any k-touching family of Jordan curves, the average distance is at most k. Theorem 5.1.8 implies that every family of Jordan curves is $6ek$-colourable, which is the bound of Fox and Pach [46] (without the $+1$). To understand the limitation of Theorem 5.1.8 it is interesting to consider the case $\alpha = o(1)$. Then δ tends to $\frac{1}{2}(1+\sqrt{5})$, and we obtain in this case that $\mathcal{F}$ is $7.14\,k$-colourable. A particular case is when $\alpha = 0$.

This is equivalent to say that any two intersecting Jordan curves are at distance 0, and therefore the family $\mathcal{F}$ of Jordan curves can be turned into a k-touching family of Jordan regions (here and everywhere else in this manuscript, it is crucial that the curves are pairwise non-crossing). Note that it was proved in [6] (see also [41]) that k-touching families of Jordan regions are $(\frac{3k}{2} + o(k))$-colourable.

In order to motivate Conjecture 5.1.7 and give it some credit, we then prove the following weaker version.

Theorem 5.1.10. *Let $\mathcal{F}$ be a family of k-touching Jordan curves. Then the average distance in $\mathcal{F}$ is at most $k/(1 + \frac{1}{16e})$.*

An immediate consequence of Theorems 5.1.8 and 5.1.10 is the following small improvement over the bound of Fox and Pach [46] in the case of Jordan curves.

Corollary 5.1.11. *Any k-touching family of Jordan curves is $15.95k$-colourable.*

An interesting connection between the chromatic number of k-touching families of Jordan curves and the packing number of directed cycles in directed planar graphs was observed by Reed and Shepherd in [92]. In a planar digraph G, let $\nu(G)$ be the maximum number of vertex-disjoint directed cycles. This quantity has a natural linear relaxation, where we seek the maximum $\nu^*(G)$ for which there are weights in $[0, 1]$ on each directed cycle of G, summing up to $\nu^*(G)$, such that for each vertex v of G, the sum of the weights of the directed cycles containing v is at most 1. It was observed by Reed and Shepherd [92] that for any G there are integers n and k such that $\nu^*(G) = \frac{n}{k}$ and G contains a collection of n pairwise non-crossing directed cycles (counted with multiplicities) such that each vertex is in at most k of the directed cycles. If we replace each directed cycle of the collection by its image in the plane, we obtain a k-touching family of Jordan curves. Assume that this family is $\beta\, k$-colourable, for some constant β. Then the family contains an independent set (a set of pairwise non-intersecting Jordan curves) of size at least $n/(\beta\, k)$. This independent set corresponds to a packing of directed cycles in G. As a consequence, $\nu(G) \geq n/(\beta\, k) = \nu^*(G)/\beta$, and then $\nu^*(G) \leq \beta\, \nu(G)$. The following is therefore a direct consequence of Corollaries 5.1.9 and 5.1.11.

Theorem 5.1.12. *For any planar directed graph G, $\nu^*(G) \leq 15.95 \cdot \nu(G)$. Moreover, if Conjecture 5.1.7 holds, then $\nu^*(G) \leq 10.22 \cdot \nu(G)$*

This improves a result of Reed and Shepherd [92], who proved that for any planar directed graph G, $\nu^*(G) \leq 28 \cdot \nu(G)$. The same result with a constant factor of 16.31 essentially followed from the result of Fox and Pach [46] (and the discussion above). Using classical results of Goemans and Williamson [50], Theorem 5.1.12 also gives improved bounds on the ratio between the maximum packing of directed cycles in planar digraphs and the dual version of the problem, namely the minimum number of vertices that needs to be removed from a planar digraph in order to obtain an acyclic digraph.

Organization of the proofs. The proofs of Theorem 5.1.2, 5.1.8 and 5.1.10 are given in Sections 5.2, 5.3 and 5.4, respectively. Section 5.5 concludes this chapter with some remarks and open problems.

5.2 Simple $k-$touching family of Jordan regions

In this section we prove Theorem 5.1.2. In the proof below we will use the following parameters instead of their numerical values (for the sake of readability): $\varepsilon = \frac{1}{4}$, $b = \frac{18}{\varepsilon} = 72$, and $k \geq 7b - 14 = 490$.

The proof proceeds by contradiction. Assume that there exists a counterexample $\mathcal{F}$, and take it with a minimum number of Jordan regions.

We will construct a bipartite planar graph G from $\mathcal{F}$ as follows: for any Jordan region d of $\mathcal{F}$ we add a vertex in the interior of d (such a vertex will be called a *disk vertex*), and for any contact point p (i.e. any point on at least two Jordan regions), we add a new vertex at p (such a vertex will be called a *contact vertex*). Now, for every Jordan region d and contact point p on d, we add an edge between the disk vertex corresponding to d and the contact vertex corresponding to p.

We now start with some remarks on the structure of G.

Claim 5.2.1. *G is a connected bipartite planar graph.*

Proof. The fact that G is planar and bipartite easily follows from the construction. If G is disconnected, then $G(\mathcal{F})$ itself is disconnected, and some connected component contradicts the minimality of $\mathcal{F}$. □

Claim 5.2.2. *All the faces of G have degree (number of edges in a boundary walk counted with multiplicity) at least 6.*

Proof. Note that by construction, the graph G is simple (i.e. there are no parallel edges). Assume for the sake of contradiction that G has a face f of degree 4. Then either f bounds three vertices (and $\mathcal{F}$ consists of two Jordan curves intersecting in a single point, in which case the theorem trivially holds), or the face f corresponds to two Jordan regions of $\mathcal{F}$ sharing two distinct points, which contradicts the fact that $\mathcal{F}$ is simple. Since G is bipartite, it follows that each face has degree at least 6. □

Two disk vertices having a common neighbor are said to be *loose neighbors* in G (this corresponds to intersecting Jordan regions in $\mathcal{F}$).

Claim 5.2.3. *Every disk vertex has at least $k + 1$ loose neighbors in G.*

Proof. Assume that some disk vertex has at most k loose neighbors in G. Then the corresponding Jordan region d of $\mathcal{F}$ intersects at most k other Jordan regions in $\mathcal{F}$. By minimality of $\mathcal{F}$, the family $\mathcal{F} \setminus \{d\}$ is $(k+1)$-colourable, and any $(k+1)$-colouring easily extends to d, since d intersects at most k other Jordan regions. We obtain a $(k + 1)$-colouring of $\mathcal{F}$, which is a contradiction. □

Claim 5.2.4. *G has minimum degree at least 2, and each contact vertex has degree at most k.*

Proof. The fact that each contact vertex has degree at least two and at most k directly follows from the definition of a k-touching family. If G contains a disk vertex v of degree at most one, then since contact vertices have degree at most k, v has at most $k - 1$ loose neighbors in G, which contradicts Claim 5.2.3. □

Claim 5.2.5. *For any edge uv, at least one of u, v has degree at least 3.*

Proof. Assume that a disk vertex u of degree 2 is adjacent to a contact vertex of degree 2. Then u has at most $1+k-1=k$ loose neighbors, which contradicts Claim 5.2.3. □

A d-vertex (resp. $\le d$-vertex, $\ge d$-vertex) is a vertex of degree d (resp. at most d, at least d). A $\ge b$-vertex is also said to be a *big vertex.* A vertex that is not big is said to be *small.*

Claim 5.2.6. *Each disk vertex of degree at most 7 has at least one big neighbor.*

Proof. Assume that some disk vertex v of degree at most 7 has no big neighbor. It follows that all the neighbors of v have degree at most $b-1$, and so v has at most $7(b-2) \le k$ loose neighbors, which contradicts Claim 5.2.3. □

We now assign to each vertex v of G a charge $\omega(v) = 2d(v) - 6$, and to each face f of G a charge $\omega(f) = d(f) - 6$ (here the function d refers to the degree of a vertex or a face). By Euler's formula, the total charge assigned to the vertices and edges of G is precisely -12. We now proceed by locally moving the charges (while preserving the total charge) until all vertices and faces have nonnegative charge. In this case we obtain that $-12 \ge 0$, which is a contradiction. The charges are locally redistributed according to the following rules (for Rule (R2), we need the following definition: a *bad vertex* is a disk 3-vertex v adjacent to two contact 2-vertices u, w, such that the three faces incident to v have degree 6 and the neighbors of u and w have degree 3).

(R1) For each big contact vertex v and each sequence of three consecutive neighbors u_1, u_2, u_3 of v in clockwise order around v, we do the following. If u_2 has a unique big neighbor (namely, v), then v gives $2-\varepsilon$ to u_2. Otherwise v gives 1 to u_2, and $(1-\varepsilon)/2$ to each of u_1 and u_3.

(R2) Each big contact vertex gives ε to each bad neighbor.

(R3) Each small contact vertex of degree at least 4 gives $\frac{1}{2}$ to each neighbor.

(R4) Each contact 3-vertex adjacent to some ≥ 3-vertex gives ε to each neighbor of degree 2.

(R5) Each disk vertex of degree at least 4 gives $1+\varepsilon$ to each neighbor of degree at most 3.

(R6) For each disk vertex v of degree 3 and each neighbor u of v with $d(u) \le 3$, we do the following. If either u has degree 3, or u has degree two and the neighbor of u distinct from v has degree at least 4, then v gives $1-\varepsilon$ to u. Otherwise, v gives 1 to u.

(R7) Each face f of degree at least 8 gives $\frac{1}{2}$ to each disk vertex incident with f.

We now analyze the new charge of each vertex and face after all these rules have been applied.

By Claim 5.2.2, all faces have degree at least 6. Since faces of degree 6 start with a charge of 0, and do not give any charge, their new charge is still 0. Let f be a face of degree $d \geq 8$. Then f starts with a charge of $d-6$ and gives at most $\frac{d}{2} \cdot \frac{1}{2}$ by Rule (R7). The new charge is then at least $d - 6 - \frac{d}{2} \cdot \frac{1}{2} = \frac{3d}{4} - 6 \geq 0$, as desired.

We now consider disk vertices. Note that these vertices receive charge by Rules (R1–4) and (R7), and give charge by Rules (R5–6). Consider first a disk vertex v of degree $d \geq 8$. Then v starts with a charge of $2d-6$ and gives at most $d(1+\varepsilon)$ (by Rule (R5)), so the new charge of v is at least $2d-6-d(1+\varepsilon) = d(1-\varepsilon)-6 \geq 0$ (since $\varepsilon = \frac{1}{4}$).

Assume now that v is a disk vertex of degree $4 \leq d \leq 7$. Then by Claim 5.2.6, v has at least one big neighbor. The vertex v starts with a charge of $2d-6$, receives at least $2-\varepsilon$ by Rule (R1), and gives at most $(d-1)(1+\varepsilon)$ by Rule (R5). The new charge of v is then at least $2d-6+2-\varepsilon-(d-1)(1+\varepsilon) = d(1-\varepsilon)-3 \geq 0$ (since $\varepsilon = \frac{1}{4}$).

We now consider a disk vertex v of degree 3. Again, it follows from Claim 5.2.6 that v has at least one big neighbor. The vertex v starts with a charge of 0, and since v has at least one big neighbor, v receives at least $2-\varepsilon$ from its big neighbors by Rule (R1). Let w be a big neighbor of v, and assume first that at least one of the two neighbors of v distinct from w (call them u_1, u_2) is not a 2-vertex adjacent to two 3-vertices. Then by Rule (R6), v gives at most $2-\varepsilon$ to u_1, u_2 (recall that by Claim 5.2.5, no two vertices of degree 2 are adjacent in G). In this case the new charge of v is at least $2-\varepsilon-(2-\varepsilon) \geq 0$, as desired. Assume now that u_1, u_2 both have degree two and their neighbors all have degree 3. In this case v gives 1 to each of u_1, u_2 and the new charge of v is at least $2-\varepsilon-2 \geq -\varepsilon$. If v is incident to a face of degree at least 8, v receives at least $\frac{1}{2}$ from such a face, and its new charge is at least $-\varepsilon + \frac{1}{2} \geq 0$, as desired. So we can assume that all the faces incident to v are faces of degree 6. In other words, v is a bad vertex. Then w gives an additional charge of ε to v by Rule (R2), and the new charge of v in this last case is at least $-\varepsilon+\varepsilon \geq 0$, as desired.

Assume now that v is a disk vertex of degree two. Then the vertex v starts with a charge of -2. By Claim 5.2.6, v has a big neighbor, call it w. By Claim 5.2.5, the neighbor of v distinct from w, call it u, has degree at least 3. If u is big then v receives a charge of $1+1=2$ by Rule (R1) and its new charge is thus at least $-2+2=0$, so we can assume that u is small (in particular, v receives $2-\varepsilon$ from w by Rule (R1)). If u has degree at least 4, then u gives a charge of $\frac{1}{2}$ to v by Rule (R3) and the new charge of v is then at least $-2+2-\varepsilon+\frac{1}{2} \geq 0$. If v lies on a face of degree at least 8, then v receives $\frac{1}{2}$ from this face by Rule (R7), and its new charge is then at least $-2+2-\varepsilon+\frac{1}{2} \geq 0$. So we can assume that u has degree 3 and all the faces containing v have degree 6. If u is adjacent to some ≥ 3-vertex, then u gives ε to v by Rule (R4), and in this case the new charge of v is at least $-2+2-\varepsilon+\varepsilon \geq 0$. So we can further assume that all the neighbors of u are 2-vertices. Call u_1, u_2 the neighbors of u distinct from v, and for $i=1,2$ let v_i be the neighbor of u_i distinct from u. Since u has degree 3, it follows from Claim 5.2.6 that v_1 and v_2 are big. Let v^+ (resp. v^-) be the neighbor of w immediately succeeding (resp. preceding) v in clockwise order around w. The faces containing v have degree 6, and since G is bipartite with minimum degree at least 2 (by Claims 5.2.1 and 5.2.4), each of these two faces is bounded by 6 vertices. As a consequence, we can assume that v^+ is adjacent to v_1 and v^- is adjacent to v_2 (see Figure 5.3). It follows that each of v^+, v^- has at least two big neighbors. Therefore, by Rule (R1), v received from w (in addition to the $2-\varepsilon$ that were taken into account earlier) $2 \cdot (1-\varepsilon)/2 = 1-\varepsilon$. So the new charge of v is at least $-2+2-\varepsilon+1-\varepsilon = 1-2\varepsilon \geq 0$, as desired.

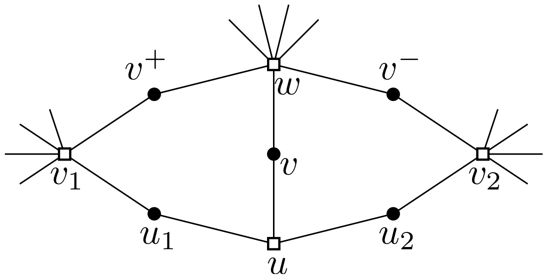

Figure 5.3: Contact vertices are depicted with white squares and disk vertices are depicted with black dots.

We now study the new charge of contact vertices. Note that contact vertices give charge by Rules (R1–4) and receive charge by Rules (R5–7). Consider a contact vertex v of degree two. Then v starts with a charge of -2. By Claim 5.2.5, the two neighbors of v (call them u an w) have degree at least 3. If they both have degree at least 4, then they both give $1+\varepsilon$ to v by Rule (R5), and the new charge of v is at least $-2+2(1+\varepsilon) \geq \varepsilon$. If one of u, w has degree at least 4 and the other has degree 3, then v receives $1+\varepsilon$ by Rule (R5) and $1-\varepsilon$ by Rule (R6). In this case the new charge of v is at least $-2+1+\varepsilon+1-\varepsilon = 0$. Finally, if u and w both have degree 3, then they both give 1 to v by Rule (R6), and the new charge of v is at least $-2+1+1=0$, as desired.

Consider a contact vertex v of degree 3. Then v starts with a charge of 0, and only gives charge if Rule (R4) applies. In this case, v gives a charge of ε to at most two of its neighbors. However, if Rule (R4) applies, then by definition, v has a neighbor of degree at least 3. Then v receives at least $1-\varepsilon$ from such a neighbor by Rules (R5–6). In this case, the new charge of v is at least $0-2\varepsilon+1-\varepsilon \geq 0$ (since $\varepsilon = \frac{1}{4}$).

Assume now that v is a contact vertex of degree $d \geq 4$. Then v starts with a charge of $2d-6$. If v is small, then v gives at most $d \cdot \frac{1}{2}$ by Rule (R3), and the new charge of v is then at least $2d-6-d\cdot\frac{1}{2} = \frac{3d}{2}-6 \geq 0$. Assume now that v is big. In this case, applications of Rule (R1) cost v no more than $d(2-\varepsilon)$ charge. We claim the following.

Claim 5.2.7. *For every big contact vertex v of degree d, applications of Rule (R2) cost v no more than $\frac{2d}{3}\cdot\varepsilon$ charge.*

Proof. We will show that v never gives a charge of ε to three consecutive neighbors of v, which implies the claim. Assume for the sake of contradiction that v gives a charge of ε to three consecutive neighbors u, w, x of v (in clockwise order around v). Assume that the neighbors of u are v, u_1, u_2 (in clockwise order around u), and the neighbors of w are v, w_1, w_2 (in clockwise order around w). Recall that by the definition of a bad vertex, each of u_1, u_2, w_1, w_2 has degree two, and all the faces incident to u or w have degree 6. Let u_1', u_2' be the neighbors of u_1, u_2 distinct from u, and let w_1', w_2' be the neighbors of w_1, w_2 distinct from w. By the definition of a bad vertex, each of u_1', u_2', w_1', w_2' has degree 3, and since all the faces incident to u or w have degree 6, $u_2' = w_1'$ and the vertices u_1', u_2', w_2' have a common neighbor, which we call y. Again,

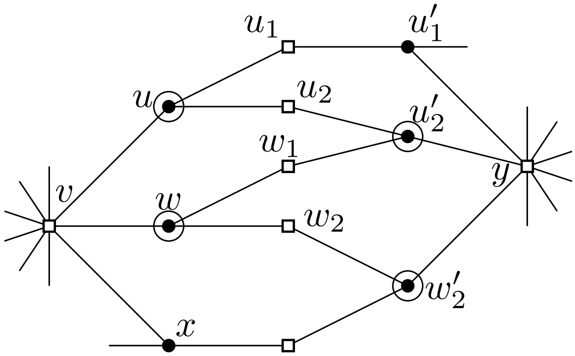

Figure 5.4: Illustration of the proof of Claim 5.2.7. The four removed vertices are circled.

by the definition of a bad vertex, the neighbor of w'_2 distinct from y has degree two and is adjacent to x (see Figure 5.4). Let $\mathcal{F}'$ be the family obtained from $\mathcal{F}$ by removing the disks corresponding to u, w, u'_2, w'_2. By minimality of $\mathcal{F}$, $\mathcal{F}'$ has a $(k+1)$-colouring c, which we seek to extend to u, w, u'_2, w'_2 (by a slight abuse of notation we identify a disk vertex of G with the corresponding disk of $\mathcal{F}$). Note that u and w'_2 have at most $k-1$ coloured neighbors, while w and u'_2 have at most $k-2$ coloured neighbors. Since $k+1$ colours are available, it follows that each of u, w'_2 has a list of at least 2 available colours, while each of w, u'_2 has a list of at least 3 available colours. We must choose a colour in each of the four lists such that each pair of vertices among u, w, u'_2, w'_2, except the pair uw'_2, are assigned different colours. This is equivalent to the following problem: take H to be the complete graph on 4 vertices minus an edge, assign to each vertex z of H an arbitrary list of at least $d_H(z)$ colours, and then choose a colour in each list such that adjacent vertices are assigned different colours. It follows from a classical result of Erdős, Rubin and Taylor [37] that this is possible for any 2-connected graph distinct from a complete graph and an odd cycle (and in particular, this holds for H). Therefore, the $(k+1)$-colouring c of $\mathcal{F}'$ can be extended to u, w, u'_2, w'_2 to obtain a $(k+1)$-colouring of $\mathcal{F}$, which is a contradiction. This proves Claim 5.2.7. $\square$

Hence, if v is a big contact vertex of degree d, then the new charge of v is at least $2d - 6 - d(2-\varepsilon) - \frac{2d}{3} \cdot \varepsilon = \frac{d}{3} \cdot \varepsilon - 6$. Since v is big, $d \geq b$ and so the new charge of v is at least $b\varepsilon/3 - 6 = 0$ (since $b = 18/\varepsilon$). It follows that the new charge of all vertices and faces is nonnegative, and then the total charge (which equals -12) is nonnegative, which is a contradiction. This concludes the proof of Theorem 5.1.2. $\square$

5.3 Chromatic number in terms of average distance

In this section we prove Theorem 5.1.8. We start with a simple lemma showing that in order to bound the chromatic number of k-touching families of Jordan curves, it is enough to bound asymptotically the number of edges in their intersection graphs.

Lemma 5.3.1. *Assume that there is a constant $a > 0$ and a function $f = o(1)$ such that for any integers k, n and any k-touching family $\mathcal{F}$ of n Jordan curves, the graph $G(\mathcal{F})$ has at most $ak(1 + f(k))n$ edges. Then for any integer k, any k-touching family of Jordan curves is $2ak$-colourable.*

Proof. Let $\mathcal{F}$ be a k-touching family of n Jordan curves, and let m denote the number of edges of $G(\mathcal{F})$. For some integer ℓ, replace each element $c \in \mathcal{F}$ by ℓ concentric copies of c, without creating any new intersection point (i.e., any portion of Jordan curve between two intersection points is replaced by ℓ parallel portions of Jordan curves). Let $\ell\mathcal{F}$ denote the resulting family. Note that $\ell\mathcal{F}$ is ℓk-touching, contains ℓn elements, and $G(\ell\mathcal{F})$ contains $\binom{\ell}{2}n+\ell^2 m$ edges. Hence, we have $\binom{\ell}{2}n+\ell^2 m < a\cdot\ell k(1+f(\ell k))\cdot\ell n$. Therefore, $m < (ak(1 + f(\ell k)) - \frac{1}{2} + \frac{1}{2\ell})n$, and $G(\mathcal{F})$ contains a vertex of degree at most $2ak(1 + f(\ell k)) - 1 + \frac{1}{\ell}$. This holds for any ℓ, and since the degree of a vertex is an integer and $f = o(1)$, $G(\mathcal{F})$ indeed contains a vertex of degree at most $2ak - 1$. We proved that k-touching families of Jordan curves are $(2ak - 1)$-degenerate, and therefore $2ak$-colourable. □

We will also need the following two lemmas.

Lemma 5.3.2. *For any integers ℓ, k, d such that $d + 2 \le \ell \le k$, and for any $p \in [0, 1)$,*

$$(1 - p)^{\ell-2} + p(1 - p)^{\ell-3}(\ell - d - 2) \ge (1 - p)^{k-2} + p(1 - p)^{k-3}(k - d - 2).$$

Proof. For fixed $d \in \mathbb{R}$ and $p \in [0, 1)$ we write $f(\ell) := (1-p)^{\ell-2}+p(1-p)^{\ell-3}(\ell-d-2)$. Note that $f(d+2) = (1-p)^d = f(d+3)$. Furthermore, for all reals $\ell \ge d+3$, $\frac{d}{d\ell}f(\ell) = (1-p)^{\ell-3}\cdot(\log(1 - p) \cdot (1 + p \cdot (\ell - d - 3)) + p) \le (1-p)^{\ell-3} (-p \cdot (1 + p \cdot (\ell - d - 3)) + p) \le 0$. So $f(\ell) \ge f(k)$ for all integers $d + 2 \le \ell \le k$. □

Lemma 5.3.3. *For any reals $1 \le \delta < 2$ and $k \ge 2$, we have $(1-\frac{\delta}{k})^{k-3} \ge (1-\frac{\delta}{k})^{k-2} \ge e^{-\delta}$.*

Proof. We clearly have $(1 - \frac{\delta}{k})^{k-3} \ge (1 - \frac{\delta}{k})^{k-2}$. To see that the second part of the inequality holds, observe first that for any real $0 \le x \le 2$, we have $e^{-x} \le 1 - \frac{x}{3}$ and thus the desired inequality holds for $k = 2, 3$.

Assume now that $k \ge 4$. Note that for any real $x \ge 0$, we have $e^{-x} \le 1 - x + \frac{x^2}{2}$. Thus,

$$\exp(-\tfrac{\delta}{k-2}) \le 1 - \tfrac{\delta}{k-2} + \tfrac{\delta^2}{2(k-2)^2} = 1 - \tfrac{\delta}{k} + \tfrac{\delta}{2k(k-2)^2}((\delta - 4)k + 8) \le 1 - \tfrac{\delta}{k},$$

with the rightmost inequality holding since $(4-\delta)k \ge 2k \ge 8$. It follows that $\exp(-\delta) \le (1 - \frac{\delta}{k})^{k-2}$, as desired. □

We are now ready to prove Theorem 5.1.8.

Proof of Theorem 5.1.8. Let $\mathcal{F}$ be a k-touching family of n Jordan curves, with average distance at most αk, and let $\delta = \delta(\alpha)$ be as provided by Theorem 5.1.8. Note that since $0 \le \alpha \le 1$, we have $1 \le \delta \le \frac{1}{2}(1+\sqrt{5}) < 2$. We denote by E the edge-set of $G(\mathcal{F})$, and by m the cardinality of E. We will prove that $m < \frac{3e^{\delta}\,k}{\delta+\delta^2(1-\alpha-\frac{2}{k})}$. Using Lemma 5.3.1, this implies that the chromatic number of any k-touching family of Jordan curves with average distance at most αk is at most $\frac{6e^{\delta}k}{\delta+\delta^2(1-\alpha)}$. Note that the chosen value $\delta(\alpha)$ of δ minimizes the value of $\frac{6e^{\delta}}{\delta+\delta^2(1-\alpha)}$. In the remainder of the proof, we will only use the fact that $1 \leq \delta < 2$.

As observed in [41], we can assume without loss of generality that each Jordan curve is a polygon (this is a simple consequence of the fact that any simple plane graph can be drawn with straight-line edges).

We recall that for two intersecting Jordan curves $a, b \in \mathcal{F}$, $\mathcal{D}(a,b)$ is the set of Jordan curves c distinct from a, b such that the (closed) region bounded by c contains exactly one of a, b, and the cardinality of $\mathcal{D}(a,b)$ (which is called *the distance between a* and b) is denoted by $d(a,b)$. For each edge $ab \in E$, we choose an arbitrary point $x(a,b)$ in the intersection of the Jordan curves corresponding to a and b. Observe that since the curves are pairwise non-crossing, $x(a,b)$ is contained in all the curves of $\mathcal{D}(a,b)$. We now select each Jordan curve of $\mathcal{F}$ uniformly at random, with probability $p = \frac{\delta}{k}$. Let $\mathcal{F}'$ be the obtained family. The expectation of the number of Jordan curves in $\mathcal{F}'$ is pn. For any pair of intersecting Jordan curves a, b, we denote by P_{ab} the probability that the set S of Jordan curves of $\mathcal{F}'$ containing $x(a,b)$ satisfies

(1) S has size at most 3,

(2) $a, b \in S$, and

(3) if $|S| = 3$, then the Jordan curve of S distinct from a and b is not an element of $\mathcal{D}(a,b)$.

Observe that

$$P_{ab} = p^2(1-p)^{\ell-2} + p^3(1-p)^{\ell-3}(\ell - d(a,b) - 2),$$

where $\ell \in \{d(a,b)+2, \ldots, k\}$ denotes the number of Jordan curves containing $x(a,b)$ in $\mathcal{F}$.

We say that an edge $ab \in E$ is *good* if a, b satisfy (1), (2), and (3) above. It follows from Lemmas 5.3.2 and 5.3.3 that the expectation of the number of good edges is

$$\begin{aligned}
\sum_{ab\in E} P_{ab} &\geq \sum_{ab\in E} \left(p^2(1-p)^{k-2} + p^3(1-p)^{k-3}(k - d(a,b) - 2)\right) \\
&\geq p^2e^{-\delta}m + p^3e^{-\delta} \sum_{ab\in E} (k - d(a,b) - 2) \\
&= p^2e^{-\delta}m \left(1 + p(k - 2 - \tfrac{1}{m} \sum_{ab\in E} d(a,b))\right). \\
&\geq p^2e^{-\delta}m \left(1 + \delta(1 - \alpha - \tfrac{2}{k})\right),
\end{aligned}$$

since $\sum_{ab\in E} d(a,b) \leq \alpha km$.

Let $\mathcal{F}''$ be obtained from $\mathcal{F}'$ by slightly modifying the Jordan curves around each intersection point x as follows. If $x = x(a,b)$, for some good edge ab, then we do the following. Note that by (1) and (2), $a, b \in \mathcal{F}'$ and $x = x(a,b)$ is contained in at most

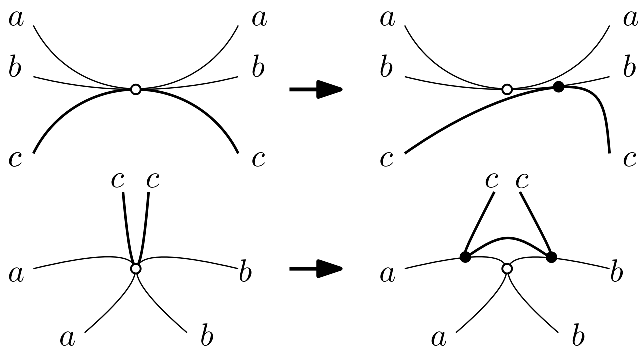

Figure 5.5: The point $x(a, b)$ is depicted by a white dot, and the newly created points are depicted by black dots.

one Jordan curve of $\mathcal{F}'$ distinct from a and b. Assume that such a Jordan curve exists, and call it c. By (3), a and b are at distance 0 in $\mathcal{F}'$. We then slightly modify c in a small disk centered in $x(a, b)$ so that for any $d \in \{a, b\}$, if c and d are at distance 0 in $\mathcal{F}'$, then they remain at distance 0 in $\mathcal{F}''$. Moreover, the point $x(a, b)$ and the newly created points are 2-touching in $\mathcal{F}''$ (see Figure 5.5). If ac (resp. bc) is also a good edge with $x(a, c) = x$ (resp. $x(b, c) = x$), then note that the conclusion above also holds with a, b replaced by a, c (resp. b, c). Now, for any other intersection point y of Jordan curves of $\mathcal{F}'$, that is not equal to $x(a, b)$ for some good edge ab, we make the Jordan curves disjoint at y. It follows from the definition of a good edge that the family $\mathcal{F}''$ obtained from $\mathcal{F}'$ after these modifications is 2-touching, and for any good edge ab, a and b are at distance 0 in $\mathcal{F}''$. Note that $G(\mathcal{F}'')$ is planar, since $\mathcal{F}''$ is 2-touching, and its expected number of edges is $\sum_{ab \in E} P_{ab}$. Since the number of edges of a planar graph is less than three times its number of vertices, we obtain:

$$3pn > \sum_{ab \in E} P_{ab} \geq p^2 e^{-\delta} m \left(1 + \delta(1 - \alpha - \tfrac{2}{k})\right).$$

As a consequence,

$$m < \frac{3e^{\delta}\, k}{\delta + \delta^2 \left(1 - \alpha - \frac{2}{k}\right)}\, n,$$

as desired. This concludes the proof of Theorem 5.1.8. □

5.4 Average distance for k−touching Jordan curves

In this section we prove Theorem 5.1.10. The following is an easy variation of the main result of Fox and Pach [46]. Consider three Jordan curves a, b, c such that a is outside the region bounded by c, b is inside the region bounded by c, and a intersects b. Then we say that the pair a, b is *c-crossing.*

Lemma 5.4.1. *Let c be a Jordan curve, and let $\mathcal{F}$ be a family of n Jordan curves such that $\mathcal{F} \cup \{c\}$ is k-touching and all the elements of $\mathcal{F}$ intersect c. Then the number of c-crossing pairs in $\mathcal{F}$ is at most $2ekn$.*

Proof. Let m be the number of c-crossing pairs in $\mathcal{F}$. For each c-crossing pair a, b in $\mathcal{F}$, we consider an arbitrary point $x(a, b)$ in $a \cap b$. We now select each Jordan curve of $\mathcal{F}$ uniformly at random with probability $p = \frac{1}{k}$. Let $\mathcal{F}'$ be the resulting family. A c-crossing pair a, b in $\mathcal{F}$ is *good* if $\mathcal{F}'$ contains a and b, but does not contain any other Jordan curve of $\mathcal{F}$ containing $x(a, b)$. Note that the probability that a given c-crossing pair a, b is good is at least $p^2(1-p)^{k-3}$, and therefore the expectation of the number of good c-crossing pairs is at least $p^2(1-p)^{k-3}m$. For any intersection point y of Jordan curves of $\mathcal{F}'$, that is not equal to $x(a, b)$ for some good c-crossing pair a, b, we make the Jordan curves disjoint at y (this is possible since the Jordan curves are pairwise non-crossing). Let $\mathcal{F}''$ be the obtained family. Observe that $\mathcal{F}''$ is 2-touching and each intersection point contains one Jordan curve lying outside the region bounded by c and one Jordan curve lying inside the region bounded by c. The graph $G(\mathcal{F}'')$ is therefore planar and bipartite. The expectation of the number of vertices of $G(\mathcal{F}'')$ is pn and the expectation of the number of edges of $G(\mathcal{F}'')$ is at least $p^2(1-p)^{k-3}m$. Since any planar bipartite graph on N vertices contains at most $2N$ edges, it follows that $p^2(1-p)^{k-3}m < 2pn$. Since $(1-\frac{1}{k})^{k-3} > e^{-1}$, we obtain that $m < 2ekn$, as desired. □

Some planar quadrangulations can be represented as 2-touching families of Jordan curves intersecting a given Jordan curve c (so that each edge of the quadrangulation corresponds to a c-crossing pair of Jordan curves). Therefore, the bound $2N$ cannot be decreased (by more than an additive constant) in the proof of Lemma 5.4.1. Furthermore, the possibly near-extremal example in Figure 5.6 shows that the bound $2ekn$ in Lemma 5.4.1 cannot be improved to less than $(2k-4)n$.

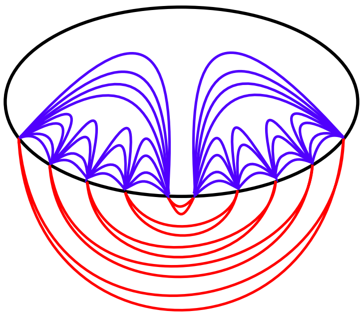

Figure 5.6: A family $\mathcal{F}$ of n Jordan curves that all intersect a fixed Jordan curve c, such that $\mathcal{F} \cup \{c\}$ is $k-$touching. Each Jordan curve in the interior of c touches each Jordan curve in the exterior of c. In the interior, there are only two sets of $k-2$ concentric Jordan curves. The remaining $n-2k+4$ Jordan curves are in the exterior of c. As n goes to infinity, the number of $c-$crossing pairs divided by n converges to $2k-4$.

We are now ready to prove Theorem 5.1.10.

Proof of Theorem 5.1.10. Let E denote the edge-set of $G(\mathcal{F})$ and let $m = |E|$. Let $\alpha = \frac{1}{km}\sum_{ab\in E} d(a,b)$. Note that the average distance in $\mathcal{F}$ is αk.

Fix some $0 < \epsilon < 1$, and set $\delta = \frac{1}{2}(1-\epsilon)$. For any edge $ab \in E$ with $d(a,b) > 0$, we do the following. Note that there is a unique ordering $c_1, \ldots, c_d$ of the elements of $\mathcal{D}(a,b)$, such that for any $1 \le i \le d$, the distance between a and c_i is $i-1$. Then the edge ab gives a charge of 1 to each of the elements $c_{\lceil \delta d\rceil}, c_{\lceil \delta d\rceil+1}, \ldots, c_{\lfloor (1-\delta)d\rfloor+1}$. Let T be the total charge given during this process. Note that

$$\begin{aligned} T &= \sum_{ab\in E} (\lfloor (1-\delta)d(a,b)\rfloor + 1 - \lceil \delta d(a,b)\rceil + 1) \\ &\ge \sum_{ab\in E} (1-2\delta)d(a,b) = \epsilon \sum_{ab\in E} d(a,b). \end{aligned}$$

We now analyze how much charge was received by an arbitrary Jordan curve c. Let $N(c)$ denote the neighborhood of c, and let $N^+(c)$ (resp. $N^-(c)$) denote the set of neighbors of c lying outside (resp. inside) the region bounded by c. Observe that if c received a charge of 1 from some edge ab, then without loss of generality we have $a \in N^+(c)$, $b \in N^-(c)$, and both a and b are at distance at most $\max(\lfloor (1-\delta)d(a,b)\rfloor, d(a,b) - \lceil \delta d(a,b)\rceil) \le (1-\delta)d(a,b) \le (1-\delta)k$ from c. Let $N_{1-\delta}(c)$ denote the set of neighbors of c that are at distance at most $(1-\delta)k$ from c. Then the charge received by c is at most the number of c-crossing pairs a,b in the subfamily of $\mathcal{F}$ induced by $N_{1-\delta}(c)$, which is at most $2ek|N_{1-\delta}(c)|$ by Lemma 5.4.1.

For any γ, let m_γ denote the number of edges $ab \in E$ such that a and b are at distance at most γk. It follows from the analysis above that $T \le 4ekm_{1-\delta}$. Therefore, $\sum_{ab\in E} d(a,b) \le \frac{4e}{\epsilon}km_{1-\delta}$. Since $\sum_{ab\in E} d(a,b) = \alpha km$, we have $m_{(1+\epsilon)/2} = m_{1-\delta} \ge \frac{\epsilon\alpha}{4e}m$.

We now study the contribution of an arbitrary edge ab to the sum $\sum_{ab\in E} d(a,b) = \alpha km$. Let t be some integer. If $d(a,b) \le \frac{t+1}{2t}k$, then ab contributes at most $\frac{t+1}{2t}k$ to αkm, and therefore at most $\frac{t+1}{2t}$ to αm. Note that there are $m_{(t+1)/2t}$ such edges ab. For each $2 \le i \le t-1$, each edge ab such that $\frac{t+i-1}{2t}k < d(a,b) \le \frac{t+i}{2t}k$ contributes at most $\frac{t+i}{2t}$ to αm, and there are $m_{(t+i)/2t} - m_{(t+i-1)/2t}$ such edges. Finally, each edge ab with $d(a,b) > \frac{2t-1}{2t}k$ contributes at most 1 to αm, and there are $m - m_{(2t-1)/2t}$ such edges. As a consequence,

$$\begin{aligned} \alpha m &\le \tfrac{t+1}{2t}m_{(t+1)/2t} + \sum_{i=2}^{t-1} \left(\tfrac{t+i}{2t}(m_{(t+i)/2t} - m_{(t+i-1)/2t})\right) + m - m_{(2t-1)/2t} \\ &= \sum_{i=1}^{t-1} \left(m_{(t+i)/2t}(\tfrac{t+i}{2t} - \tfrac{t+i+1}{2t})\right) + m \\ &= m - \tfrac{1}{2t}\sum_{i=1}^{t-1} m_{(t+i)/2t} \\ &\le m - \tfrac{1}{2t}\sum_{i=1}^{t-1} \tfrac{i}{t}\tfrac{\alpha}{4e}m, \end{aligned}$$

since $m_{(1+\epsilon)/2} \geq \frac{\epsilon\alpha}{4e} m$ for every $0 < \epsilon < 1$. As a consequence, we obtain that $\alpha \leq 1 - \frac{t-1}{t}\frac{\alpha}{16e}$. Since this holds for any integer t, we have $\alpha \leq 1 - \frac{\alpha}{16e}$ and therefore $\alpha \leq 1/(1+\frac{1}{16e})$, as desired. □

5.5 Remarks and open questions

Most of the proof of Theorem 5.1.2 proceeds by finding a Jordan region intersecting at most k other Jordan regions (see Claim 5.2.3). On a single occasion, we use a different reduction (via a list-colouring argument). A natural question is: could this be avoided? Is it true that in any simple k-touching family of Jordan regions, if k is large enough, then there is a Jordan region which intersects at most k other Jordan regions? It turns out to be wrong, as depicted in Figure 5.7. However, a proof along the lines of that of Theorem 5.1.2 (but significantly simpler), shows that if k is large enough, then there is a Jordan region which intersects at most $k+1$ other Jordan regions. It was pointed out to us by Patrice Ossona de Mendez (after the original version of this manuscript was submitted) that he also obtained this result in 1999 (see [89]). His result and its proof are stated with a completely different terminology, but the ideas are essentially the same. In particular, his result also implies (relatives of) our Corollaries 5.1.3 and 5.1.4.

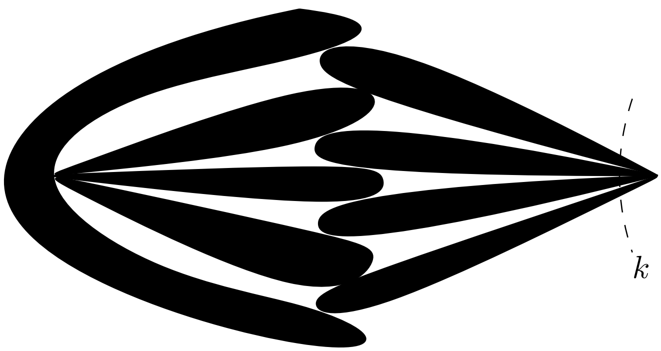

Figure 5.7: Every Jordan region intersects precisely $k+1$ other Jordan regions.

This can be used to obtain a result on the chromatic number of *simple* families of k-touching Jordan curves (families of k-touching Jordan curves such that any two Jordan curves intersect in at most one point). Using the result mentioned above (that if k is sufficiently large and the interiors are pairwise disjoint, then there is a Jordan region that intersects at most $k+1$ other Jordan regions), it is not difficult to show that the chromatic number of any simple family of k-touching Jordan curves is at most $2k$ plus a constant. We believe that the answer should be much smaller.

Problem 5.5.1. *Is it true that for some constant c, any simple family of k-touching Jordan curves can be coloured with at most $k+c$ colours?*

It was conjectured in [41] that if $\mathcal{S}$ is a family of pairwise non-crossing strings such that (i) any two strings intersect in at most one point and (ii) any point of the plane is on at most k strings, then $\mathcal{S}$ is $(k + c)$-colourable, for some constant c. Note that, if true, this conjecture would give a positive answer to Problem 5.5.1.

Chapter 6

The dimension of the Incipient Infinite Cluster.

In this chapter, we study the Incipient Infinite Cluster (IIC) of high-dimensional bond percolation on $\mathbb{Z}^d$. We prove that the mass dimension of IIC almost surely equals 4 and the volume growth exponent of IIC almost surely equals 2.

6.1 Introduction

Consider critical nearest-neighbour percolation on $\mathbb{Z}^d$. The *Incipient Infinite Cluster* (IIC) is a random infinite subset of $\mathbb{Z}^d$ which intuitively can be viewed as the critical cluster of the origin, conditioned to be infinitely large. This conditioning induces a new probability measure $\mathbb{P}_{\rm IIC}$. We study the IIC in *high dimensions d* (see below for formal definitions) and in particular we identify the typical size of IIC under $\mathbb{P}_{\rm IIC}$. In order to sensibly determine the size of the IIC we use the concepts of *mass dimension* $d_m(A)$ of a subset $A \subset \mathbb{Z}^d$ and the *volume growth exponent* $d_f(G)$ of an infinite connected graph G. The former measures the IIC with respect to the (extrinsic) distance of the space $\mathbb{Z}^d$ in which IIC is embedded, while the latter measures the induced graph of IIC with respect to (intrinsic) graph distance. We prove that the mass dimension of IIC is 4 and the volume growth exponent of the graph of IIC is 2, $\mathbb{P}_{\rm IIC}$-almost surely. See Theorems 1 and 2 below. Theorem 1 gives an explicit and rigorous foundation for the intuition that for high d the IIC is a 4-dimensional object, a conjecture of physicists going back at least 30 years [2][4].

6.1.1 Critical high-dimensional bond percolation

Let $G = (\mathbb{Z}^d, E)$ be a graph and fix a parameter $p \in [0, 1]$. We focus on the case of *nearest-neighbour* bond percolation, meaning that $(x, y) \in E \Leftrightarrow \|x - y\|_1 = 1$ and each edge (also called bond) $e \in E$ is independently declared *open* with probability p and *closed* with probability $1 - p$. Here $\|x\|_1$ denotes the ℓ_1-norm of $x \in \mathbb{Z}^d$. The resulting probability measure is denoted by $\mathbb{P}_p$.

Let $\{x \leftrightarrow y\}$ denote the event that vertices x and y are connected by a finite path of open edges. Let $\mathscr{C}(x) = \{y \in \mathbb{Z}^d \mid x \leftrightarrow y\}$ denote the *open cluster* of x. It is well known that for $d \geq 2$ there exists a critical probability $p_c \in (0,1)$ for which the model undergoes a phase transition:

$$\mathbb{P}_{p_c}(\exists x \in \mathbb{Z}^d \ s.t. \ |\mathscr{C}(x)| = \infty) = \begin{cases} 0 & \text{if } p < p_c; \\ 1 & \text{if } p > p_c. \end{cases} \tag{6.1}$$

Later we will zoom in on what happens at $p = p_c$. Let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{Z}^d$. This choice of norm is not essential, since all norms on $\mathbb{Z}^d$ are equivalent and we only work with estimates that hold up to a constant value. For functions f and g, we let $f \asymp g$ denote that $cg \leq f \leq Cg$ holds asymptotically for some constants $c, C > 0$. Throughout this chapter we assume that our lattice is *high-dimensional*, by which we mean that $d > 6$ is such that

$$\mathbb{P}_{p_c}(x \leftrightarrow y) \asymp \|x - y\|^{2-d}, \tag{6.2}$$

for $x, y \in \mathbb{Z}^d$. It is widely believed that (6.2) holds in all dimensions $d > 6$. In case of nearest-neighbour percolation it has been known for some time that (6.2) is true for all $d \geq 19$ [55] and recently V.d. Hofstad and Fitzner [45] proved it for $d \geq 11$. If there exists an $L > 0$ such that $(x, y) \in E \Leftrightarrow \|x - y\| \leq L$, then we speak of *spread-out finite-range* percolation, rather than nearest-neighbour percolation. For this model, it has been proven that (6.2) holds in $d > 6$ if the lattice is sufficiently spread out, which means that L should be large enough [58]. For readability we restrict ourselves to the case of nearest-neighbour percolation, but all results in this chapter also hold for spread-out finite-range percolation.

In the regime of high dimensions, calculations are relatively easy. In technical practice this is often a consequence of validity of the bound (6.2) on the two-point function, but the intuitive idea behind all this is that for d larger than a certain critical dimension d_c, of which the value is believed to be 6, the model attains *mean-field* behaviour. The amount of space in which open paths can travel has become so large that different pieces of a critical cluster hardly interact. In particular, large open cycles have very small probability. Therefore an open cluster will for many questions behave like a connected graph without cycles: a tree. Percolation on a tree is relatively easy.

Incipient Infinite Cluster We now focus on what happens during the phase transition at $p = p_c$. In particular, we want to know how critical clusters behave 'as they are becoming infinitely large'. This interpretation is the source of the name *Incipient Infinite Cluster* (IIC), a term originating from the physics literature, which was first defined and treated in a mathematically rigorous way by Kesten [69]. See below for a formal definition.

It turns out that $\mathbb{P}_{p_c}(|\mathscr{C}(0)| = \infty) = 0$ in high dimensions [56], so working directly with $\mathbb{P}_{p_c}$ will not provide us with interesting detailed information about an infinite cluster. This problem can be overcome by conditioning on some event that implies that $|\mathscr{C}(0)| = \infty$, thus constructing a new probability measure. There exist several constructions of such an IIC-measure that have been proven to be equivalent, providing evidence that the IIC is quite a canonical, robust and unique object. For a precise characterization, the reader is referred to [59] and [100]. We will only directly need the following construction:

$$\mathbb{P}_{\mathrm{IIC}}(F) = \lim_{\|x\|\to\infty} \mathbb{P}_{p_c}(F \mid 0 \leftrightarrow x) \tag{6.3}$$

for cylinder events F. In high dimensions, the limit exists irrespective of the direction. Through references to literature we will also implicitly use the construction

$$\mathbb{Q}_{\mathrm{IIC}}(F) = \lim_{p\uparrow p_c} \frac{\sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\left(F \cap \{0 \leftrightarrow x\}\right)}{\sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\left(0 \leftrightarrow x\right)}.$$

In high dimensions, the limits $\mathbb{P}_{\mathrm{IIC}}(F)$ and $\mathbb{Q}_{\mathrm{IIC}}(F)$ exist and are equal for all cylinder events F. Consequently $\mathbb{P}_{\mathrm{IIC}}$ and $\mathbb{Q}_{\mathrm{IIC}}$ extend to the same probability measure in our context [59],[100]. Expectation value with respect to $\mathbb{P}_{\mathrm{IIC}}$ is denoted by $\mathbb{E}_{\mathrm{IIC}}$. It holds that $\mathbb{P}_{\mathrm{IIC}}\left(|\mathscr{C}(0)| = \infty\right) = 1$ and partly because of this, some authors refer to the IIC as the distribution of $\mathscr{C}(0)$ under $\mathbb{P}_{\mathrm{IIC}}$. However, in the context of $\mathbb{P}_{\mathrm{IIC}}$ the term IIC is also often used to refer to the infinite cluster at the origin itself. We adopt the latter convention.

Definition. *IIC is the random graph with vertex set $\mathscr{C}(0)$ and induced edge set*

$$\{(x,y) \in \mathscr{C}(0) \times \mathscr{C}(0) \mid (x,y) \text{ is open }\}.$$

In many cases we are only interested in the vertices and therefore we abuse notation by writing IIC $= \mathscr{C}(0)$.

6.1.2 Mass dimension and volume growth exponent

In order to determine how large the (infinite) IIC is, we need to associate some natural notion of dimensionality. On the one hand, we will calculate the mass dimension, which counts the vertices of IIC that are in a cube of finite radius r around the origin. On the other hand, we consider the volume growth exponent, which counts the number of vertices in IIC that can be reached from the origin by an open path of length at most some fixed r. In the former case, IIC is counted with respect to the 'extrinsic' (Euclidean) metric of the underlying lattice $\mathbb{Z}^d$, while in the latter case, IIC is counted with respect to the 'intrinsic' graph distance of the random graph.

Auxiliary definitions. *Denote by*

$$Q_r = \left\{x \in \mathbb{Z}^d \mid \|x\| \le r\right\}$$

the cube with radius r and boundary

$$\partial Q_r = Q_r \backslash Q_{r-1}.$$

In practice we will want to bound the cardinality of the following three random sets,

$$X_r = \{x \in Q_r \mid 0 \leftrightarrow x\}$$

$$X_{r,r} = \left\{x \in Q_r \mid 0 \xleftrightarrow{Q_r} x\right\}$$

$$B_r = \left\{ x \in \mathbb{Z}^d \mid 0 \overset{\leq r}{\longleftrightarrow} x \right\}$$

where $0 \overset{Q_r}{\longleftrightarrow} x$ *means that* 0 *is connected to* x *by an open path that does not leave* Q_r *and* $0 \overset{\leq r}{\longleftrightarrow} x$ *means that* 0 *is connected to* x *by an open path of length* $\leq r$ *(with respect to graph distance in the random percolated graph).*

Definition of dimensions. *The mass dimension of a subset* $A \subset \mathbb{Z}^d$ *is*

$$d_m(A) = \lim_{r\to\infty} \log_r |A \cap Q_r|$$

if the limit exists. The volume growth exponent of an infinite connected graph G *is defined by*

$$d_f(G) = \lim_{r\to\infty} \log_r |B_G(x,r)|$$

if the limit exists. Here $B_G(x,r)$ *is the ball with some center vertex* x *and radius* r*, with respect to graph distance.*

Note that the mass dimension of IIC equals $d_m(\text{IIC}) = \lim_{r\to\infty} \log_r |X_r|$ and the volume growth exponent of IIC can be rewritten as $d_f(\text{IIC}) = \lim_{r\to\infty} \log_r |B_{\text{IIC}}(0,r)| = \lim_{r\to\infty} \log_r |B_r|$.

Our main goal is to prove Theorem 6.1.1, which states that on a high-dimensional lattice the mass dimension of IIC almost surely equals 4.

Theorem 6.1.1. *In high dimensions,*

$$\mathbb{P}_{IIC}\left(d_m(IIC) \equiv \lim_{r\to\infty} (\log_r |X_r|) = 4\right) = 1.$$

This can be contrasted against Theorem 6.1.2, which states that on a high-dimensional lattice the volume growth exponent of IIC almost surely equals 2. This second result was already implicit in two auxiliary lemmas in [78], which we use to obtain a formal derivation of the almost sure statement.

Theorem 6.1.2. *In high dimensions,*

$$\mathbb{P}_{IIC}\left(d_f(IIC) \equiv \lim_{r\to\infty} (\log_r |B_r|) = 2\right) = 1.$$

6.1.3 Embedding and conjectures

On the 4-dimensionality of IIC. Earlier developments in the direction of determining 'the' dimension of the IIC include the following. In [100] it was shown that in high dimensions, $\mathbb{P}_{\text{IIC}}(0 \leftrightarrow x) \asymp \|x\|^{4-d}$, implying that $\mathbb{E}_{\text{IIC}}(|X_r|) \asymp C \cdot r^4$. This moment bound, which is also derived in a more general setting in [59], already gave some weak notion of the 4-dimensionality of the IIC. As we will see later, it provides enough information to derive an almost sure upper bound 4 on the (upper) mass dimension of IIC, essentially using Markov's inequaliy and Borel-Cantelli. However, deriving the corresponding lower bound 4 on the (lower) mass dimension requires a completely different technique. Concentration inequalities like the second moment method are not powerfull enough [17] and many standard techniques from percolation theory don't apply because of the delicate dependency on the origin, induced by the IIC-measure. Indeed, the derivation of the lower bound constitutes the main contribution of this chapter.

Spectral dimension and other bounds on $|X_r|$ and $|B_r|$. The spectral dimension of an infinite connected graph G is defined by

$$d_s(G) = -2 \cdot \lim_{r\to\infty} \log_r\left(p_{2r}(x,x)\right)$$

if the limit exists. Here $p_{2r}(x,x)$ is the return probability of a simple random walk on G after r steps. Kozma and Nachmias [78] showed that $d_s(\text{IIC}) = \frac{4}{3}$, thereby confirming the so-called Alexander-Orbach conjecture in high dimensions. For many 'nice' graphs and in particular for any Cayley graph G it holds that $d_f(G) = d_s(G)$, but this is not the case for the IIC, as $d_f(\text{IIC}) = 2 \neq \frac{4}{3} = d_s(\text{IIC})$, suggesting that the IIC is an intrinsically fractal object. Kozma and Nachmias also showed that $\mathbb{E}_{p_c}\left(|B_r|\right) \asymp r$ and $\mathbb{P}_{p_c}\left(B_r \backslash B_{r-1} \neq \emptyset\right) \asymp r^{-1}$. These statements are in terms of the intrinsic graph distance and should be contrasted against their extrinsic counterparts $\mathbb{E}_{p_c}\left(|X_r|\right) \asymp r^2$ and $\mathbb{P}_{p_c}\left(0 \leftrightarrow \partial Q_r\right) \asymp r^{-2}$ [59][79].

Growth behaviour of the boundary of $X_{r,r}$. In the proof of Theorem 6.1.1, we actually also show that $\mathbb{P}_{\text{IIC}}\left(\lim_{r\to\infty} \log_r(|X_{r,r}|) = 4\right) = 1$. That is, $|X_{r,r}|$ and $|X_r|$ don't differ very much; they both grow like r^4. Define the boundary $\partial X_r := \{x \in \partial Q_r \mid 0 \leftrightarrow x\}$. Since $X_r = \bigsqcup_{k=1}^r \partial X_k$, it is to be expected that $|\partial X_r|$ typically grows like r^3. Similarly, if we define the 'boundary' $\partial X_{k,r} := \left\{x \in \partial Q_k \mid 0 \overset{Q_r}{\longleftrightarrow} x\right\}$ then $X_{r,r} = \bigsqcup_{k=1}^r \partial X_{k,r}$, so one would expect that $|\partial X_{k,r}|$ grows like k^3. We believe this is indeed the case for $k \ll r$, because for those values $|\partial X_{k,r}| \approx |\partial X_k|$. However, if $k \approx r$ the picture (presumably) changes completely. Theorem 1.16 in [17] yields that there exists a constant $C > 0$ such that for all $\lambda, r > 0$, $\mathbb{P}_{\text{IIC}}\left(\sum_{k=1}^r |\partial X_{k,k}| \leq \frac{1}{\lambda} \cdot r^3\right) \leq C \cdot \frac{1}{\lambda}$. A slight adaptation of that proof yields that $\mathbb{P}_{\text{IIC}}\left(|\partial X_{r,r}| \leq \frac{1}{\lambda} \cdot r^2\right) \leq C \cdot \frac{1}{\lambda}$ and in fact, we conjecture that the opposite bound $\mathbb{P}_{\text{IIC}}\left(|\partial X_{r,r}| \geq \lambda \cdot r^2\right) \leq C \cdot \frac{1}{\lambda}$ holds too. In other words, we expect $|\partial X_{r,r}|$ to grow like r^2 instead of r^3. One motivation for the opposite bound comes from Theorem 2 in [79], which essentially says that $|X_{r,r}|$ is smaller than r^2 if $|X_r|$ is smaller than r^4. To actually prove the opposite bound, it would suffice to show that $\mathbb{E}_{\text{IIC}}\left(|\partial X_{r,r}|\right) \leq C \cdot r^2$, and for this it would be very useful to have a good upper bound on $\mathbb{P}_{\text{IIC}}(0 \overset{Q_r}{\longleftrightarrow} x)$, for $x \in \partial Q_r$. While $\mathbb{P}_{\text{IIC}}(0 \longleftrightarrow x) \asymp \|x\|^{4-d}$ depends only on the norm of $\|x\|$ but not really on the choice of norm, the behaviour of $\mathbb{P}_{\text{IIC}}(0 \overset{Q_r}{\longleftrightarrow} x)$ is more complicated. For example, if we define the cube Q_r with respect to the ℓ_∞-norm, then it is much 'harder' for an open path that stays entirely inside Q_r to reach a corner vertex x_1 of Q_r, than it is to reach the center vertex x_2 of a face of Q_r, although $\|x_1\|_\infty = \|x_2\|_\infty$.

The backbone of IIC and scaling limits. There is a natural subset of the IIC, called the *backbone* (bb) of the IIC, which consists of all open bonds $e = (e_-, e_+)$ such that there exist two disjoint open paths, one path from 0 to e_- and the other path from e_+ to ∞. It is expected that the mass dimension of the backbone $\mathbb{P}_{\text{IIC}}$-almost surely equals 2. The validity of the almost sure upper bound 2 is immediate from the known expectation bound $\mathbb{E}_{\text{IIC}}\left(|bb \cap Q_r|\right) \asymp r^2$ [59] and an application of Lemma 6.3.2. Heydenreich, V.d. Hofstad, Hulshof and Miermont prepare a proof that the scaling limit of the backbone is a brownian motion, which almost surely has Hausdorff dimension 2. A related, but wide open conjecture is that the scaling limit of the high-dimensional IIC itself is *Integrated super-Brownian excursion* [57].

The IIC in low dimensions. For $d = 1$, IIC trivially has mass dimension and volume growth exponent 1. Kesten proved the bound

$$\mathbb{E}_{\mathrm{IIC}}|\mathrm{IIC} \cap Q_r| \asymp r^2 \cdot \mathbb{P}_{p_c}(0 \leftrightarrow \partial Q_r),$$

which holds for a wide range of lattices on $\mathbb{Z}^2$ [69]. For site percolation on the triangular lattice, Lawler, Schramm and Werner were able to show that $\mathbb{P}_{p_c}(0 \leftrightarrow \partial Q_r) = r^{-5/48+o(1)}$ [80]. So for this particular lattice, $\mathbb{E}_{\mathrm{IIC}}|\mathrm{IIC} \cap Q_r| \asymp r^2 \cdot r^{-5/48} = r^{91/48}$. By the conjectured universality of the exponent, this result presumably holds for all common two-dimensional lattices. Note that $\frac{91}{48}$ is just slightly smaller than 2, the dimension of the surrounding space. For $3 \leq d \leq 6$ very little is known rigorously. Simulations by Kumagai suggest that $d_s(\mathrm{IIC})$ ranges from $\approx 1.318+/-0.001$ for $d = 2$ to $\approx 1.34 + / - 0.02$ for $d = 5$, which is close to the value $4/3$ that holds in high dimensions, but nevertheless supports the belief that the Alexander-Orbach conjecture is false for $d \leq 6$ [78].

6.1.4 About the proof

For Theorem 1 we use an upper bound on the expectation value of $|X_r|$ to derive that $d_m(\mathrm{IIC}) \leq 4$, almost surely. The lower bound is the hard (or at least more unusual) part. For this we use the one-arm exponent bound $\mathbb{P}_{p_c}(0 \leftrightarrow \partial Q_r) \leq C \cdot \frac{1}{r^2}$, from which it will follow that under $\mathbb{P}_{\mathrm{IIC}}$ a typical shortest open path between 0 and ∂Q_r has length r^2. In Theorem 6.3.1 this is combined with the fact that the intrinsic ball B_r contains approximately r^2 elements, yielding that $|X_r| \geq |X_{r,r}| \approx |B_{(\text{length shortest open path } 0 \leftrightarrow \partial Q_r)}| \approx |B_{r^2}| \approx (r^2)^2 = r^4$, or rather that large downwards deviations of these approximations have small enough probability. The workhorse of this chapter is Lemma 6.3.2, which turns probabilistic bounds into almost sure statements. Indeed, Theorem 6.1.2 follows by a direct application of this lemma to a result from literature.

6.2 Ingredients from literature

In this section we collect ingredients from the literature that we use in our proofs.

Theorem 6.2.1 (Theorem 1.5 in [59])**.** *In high dimensions, there exists a constant $C > 0$ such that for all $r \geq 1$:*

$$\mathbb{E}_{IIC}(|X_r|) \leq C \cdot r^4.$$

Theorem 6.2.2 (Corollary of Theorem 1 in [79])**.** *In high dimensions, there exists a $C > 0$ such that for all $r \geq 1$:*

$$\mathbb{P}_{p_c}(0 \leftrightarrow \partial Q_r) \leq C \cdot \frac{1}{r^2}.$$

Lemma 6.2.3 (Lemma 2.5 in [78]). *In high dimensions, there exists a constant $C > 0$ such that for all $r \geq 1$ and any event E measurable with respect to B_r and for any $x \in \mathbb{Z}^d$ with $\|x\|$ sufficiently large:*

$$\mathbb{P}_{p_c}(E \cap \{0 \leftrightarrow x\}) \leq C \cdot \sqrt{r \cdot \mathbb{P}_{p_c}(E)} \cdot \mathbb{P}_{p_c}(0 \leftrightarrow x).$$

Lemma 6.2.4 (Essentially Lemma 6.1 in [101]). *In high dimensions, there exists a $C > 0$ such that for all $\epsilon > 0, r \geq 1$:*

$$\mathbb{P}_{IIC}\left(0 \overset{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right) \leq C \cdot \sqrt{\epsilon},$$

where $\left\{0 \overset{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right\}$ *is the event that* 0 *is connected to* ∂Q_r *by an open path of length* $\leq \epsilon \cdot r^2$.

Proof. The event $E = \left\{0 \overset{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right\}$ is measurable with respect to $B_{\epsilon \cdot r^2}$. Therefore, Lemma 6.2.3 implies that for any $x \in \mathbb{Z}^d$ with $\|x\|$ sufficiently large,

$$\mathbb{P}_{p_c}\left(0 \overset{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r \mid 0 \leftrightarrow x\right) \leq C' \cdot \sqrt{\epsilon \cdot r^2 \cdot \mathbb{P}_{p_c}(0 \leftrightarrow \partial Q_r)} \leq C \cdot \sqrt{\epsilon},$$

where the second inequality follows from Theorem 6.2.2. Now apply construction (6.3) of $\mathbb{P}_{\mathrm{IIC}}$. □

Lemma 6.2.5 (Essentially Lemmas 2.2 and 2.3 in [78]). *In high dimensions, there exists a $C > 0$ such that for all $\lambda > 1$ and $r \geq 1$:*

$$\mathbb{P}_{IIC}\left(|B_r| \leq \frac{1}{\lambda} \cdot r^2\right) \leq C \cdot \frac{1}{\lambda} \tag{6.4}$$

and

$$\mathbb{P}_{IIC}\left(|B_r| \geq \lambda \cdot r^2\right) \leq C \cdot \frac{1}{\lambda}. \tag{6.5}$$

Proof. Inequality (6.4) is the statement of Lemma 2.3 in [78]. On the other hand, Lemma 2.2 in [78] states that there exists a $C > 0$ such that for all $r \geq 1$ and all $x \in \mathbb{Z}^d$ with $\|x\|$ sufficiently large,

$$\mathbb{E}_{p_c}\left(|B_r| \cdot \mathbb{1}_{\{0 \leftrightarrow x\}}\right) \leq C \cdot r^2 \cdot \mathbb{P}_{p_c}(0 \leftrightarrow x).$$

By Markov's inequality this implies that for all $\lambda > 1$ and $r \geq 1$ it holds that $\mathbb{P}_{p_c}\left(|B_r| \geq \lambda \cdot r^2 \mid 0 \leftrightarrow x\right) \leq C \cdot \frac{1}{\lambda}$, for all $x \in \mathbb{Z}^d$ with $\|x\|$ sufficiently large. Letting $\|x\| \to \infty$ yields (6.5), because $\left\{|B_r| \geq \lambda \cdot r^2\right\}$ is a cylinder event. □

6.3 Deriving the main theorems

The following theorem is crucial for the derivation of Theorem 6.1.1. It relies on Lemmas 6.2.4 and 6.2.5 and in that sense, it uses that both the cardinality of the intrinsic ball with radius r and the length of the shortest path from 0 to the boundary of ∂Q_r grow like r^2.

Theorem 6.3.1. *In high dimensions, there exists a $C > 0$ such that for all $\lambda > 1$ and $r \geq 1$:*

$$\mathbb{P}_{IIC}\left(|X_{r,r}| \leq \frac{1}{\lambda} \cdot r^4\right) \leq C \cdot \frac{1}{\lambda^{1/5}}.$$

Proof. Let $\lambda > 1$. Write $\epsilon := \epsilon(\lambda) = \lambda^{-2/5}$, then

$$\begin{aligned}\mathbb{P}_{\mathrm{IIC}}\left(|X_{r,r}| \leq \frac{1}{\lambda} \cdot r^4\right) &= \mathbb{P}_{\mathrm{IIC}}\left(|X_{r,r}| \leq \frac{1}{\lambda} \cdot r^4, 0 \stackrel{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right) + \\ &\quad \mathbb{P}_{\mathrm{IIC}}\left(|X_{r,r}| \leq \frac{1}{\lambda} \cdot r^4, \text{ not } 0 \stackrel{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right). \qquad (6.6)\end{aligned}$$

By Lemma 6.2.4 we can bound the first term as follows:

$$\mathbb{P}_{\mathrm{IIC}}\left(|X_{r,r}| \leq \frac{1}{\lambda} \cdot r^4, 0 \stackrel{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right) \leq \mathbb{P}_{\mathrm{IIC}}\left(0 \stackrel{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right) \leq C \cdot \epsilon^{1/2} = C \cdot \frac{1}{\lambda^{1/5}}. \quad (6.7)$$

On the other hand, if $\left\{\text{not } 0 \stackrel{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right\}$ occurs then the intrinsic ball $B_{\epsilon \cdot r^2}$ is a subset of $X_{r,r}$, so $|B_{\epsilon \cdot r^2}| \leq |X_{r,r}|$. Therefore a bound on the second term is given by

$$\begin{aligned}\mathbb{P}_{\mathrm{IIC}}\left(|X_{r,r}| \leq \frac{1}{\lambda} \cdot r^4, \text{ not } 0 \stackrel{\leq \epsilon \cdot r^2}{\longleftrightarrow} \partial Q_r\right) &\leq \mathbb{P}_{\mathrm{IIC}}\left(|B_{\epsilon \cdot r^2}| \leq \frac{1}{\lambda} \cdot r^4\right) \\ &= \mathbb{P}_{\mathrm{IIC}}\left(|B_{\epsilon \cdot r^2}| \leq \frac{1}{\lambda \cdot \epsilon^2} \cdot (\epsilon \cdot r^2)^2\right) \\ &\leq C \cdot \frac{1}{\lambda \cdot \epsilon^2} \\ &= C \cdot \frac{1}{\lambda^{1/5}}, \qquad (6.8)\end{aligned}$$

where the second inequality follows from Lemma 6.2.5. Now evaluate (6.7) and (6.8) in (6.6) to finish the proof. □

The next lemma will be used to transform the results obtained so far into the almost sure statements of Theorem 6.1.1 and 6.1.2. We present a more general and stronger version than we actually need.

Lemma 6.3.2. *Let $Z_1, Z_2, \ldots$ be a sequence of random variables with values in $\mathbb{R}_{>0}$, such that $Z_1 \leq Z_2 \leq \ldots$*

1. *If there exist constants $\beta, \mu, C > 0$ such that at least one of the following two conditions holds*

- $\mathbb{E}(Z_r) \leq C \cdot r^{\beta}$ *for all* $r > 0$*;*
- $\mathbb{P}(Z_r \geq \lambda \cdot r^{\beta}) \leq C \cdot \frac{1}{\log(\lambda)^{1+\mu}}$ *for all* $\lambda > 1$ *and* $r > 0$*,*

then:

$$\mathbb{P}\left(\limsup_{r\to\infty}\left(\log_r(Z_r)\right) \leq \beta\right) = 1. \tag{6.9}$$

2. *If there exist constants* $\alpha, \mu, C > 0$ *such that at least one of the following two conditions holds*
   - $\mathbb{E}\left(\frac{1}{Z_r}\right) \leq C \cdot r^{-\alpha}$ *for all* $r > 0$*;*
   - $\mathbb{P}(Z_r \leq \frac{1}{\lambda} \cdot r^{\alpha}) \leq C \cdot \frac{1}{\log(\lambda)^{1+\mu}}$ *for all* $\lambda > 1$ *and* $r > 0$*,*

   then:

$$\mathbb{P}\left(\liminf_{r\to\infty}\left(\log_r(Z_r)\right) \geq \alpha\right) = 1. \tag{6.10}$$

Proof. First note that the first condition of (6.9) implies the second condition of (6.9). Indeed, by Markov's inequality there exist $C, \mu > 0$ such that for all $\lambda > 1$ and $r > 0$

$$\mathbb{P}(Z_r \geq \lambda \cdot r^{\beta}) \leq \frac{\mathbb{E}(Z_r)}{\lambda \cdot r^{\beta}} \leq \frac{C \cdot r^{\beta}}{\lambda \cdot r^{\beta}} \leq C \cdot \frac{1}{\log(\lambda)^{1+\mu}}.$$

Similarly, the first condition of (6.10) implies the second condition of (6.10). Indeed,

$$\mathbb{P}\left(Z_r \leq \frac{1}{\lambda} \cdot r^{\alpha}\right) = \mathbb{P}\left(\frac{1}{Z_r} \geq \lambda \cdot r^{-\alpha}\right) \leq \frac{\mathbb{E}\left(\frac{1}{Z_r}\right)}{\lambda \cdot r^{-\alpha}} \leq \frac{C \cdot r^{-\alpha}}{\lambda \cdot r^{-\alpha}} \leq C \cdot \frac{1}{\log(\lambda)^{1+\mu}}.$$

It remains to prove (6.9) and (6.10) under their second condition.

Define the strictly increasing subsequences $r_k = 2^k$ and $\lambda_k = 2^{\left(k^{\left(\frac{1+\mu/2}{1+\mu}\right)}\right)}$. Also define $\epsilon_k := \log_{r_k}(\lambda_k) = k^{\left(\frac{1+\mu/2}{1+\mu}-1\right)}$. Note that $r_k, \epsilon_k > 0$ and $\lambda_k > 1$ for all positive integers k, and $\lim_{k\to\infty} \epsilon_k = 0$. We first prove (6.9). For all positive integers k it holds that

$$\mathbb{P}\left(Z_{r_k} \geq \lambda_k \cdot r_k^{\beta}\right) \leq C \cdot \frac{1}{\log(\lambda_k)^{1+\mu}}. \tag{6.11}$$

Using the notation $Y_r := \log_r(Z_r)$ we obtain that

$$\begin{aligned}
\sum_{k=1}^{\infty} \mathbb{P}\left(Y_{r_k} \geq \epsilon_k + \beta\right) &= \sum_{k=1}^{\infty} \mathbb{P}\left(Z_{r_k} \geq \lambda_k \cdot r_k^{\beta}\right) \\
&\leq C \cdot \sum_{k=1}^{\infty} \frac{1}{\log(\lambda_k)^{1+\mu}} \\
&= \frac{C}{\log(2)^{1+\mu}} \cdot \sum_{k=1}^{\infty} \frac{1}{k^{1+\mu/2}} \\
&< \infty.
\end{aligned}$$

By Borel-Cantelli this implies that

$$\mathbb{P}\left(Y_{r_k} \geq \epsilon_k + \beta \text{ for infinitely many } k\right) = 0$$

and because $\lim_{k\to\infty} \epsilon_k = 0$ it follows that

$$\mathbb{P}\left(\limsup_{k\to\infty}(Y_{r_k}) \leq \beta\right) = 1. \tag{6.12}$$

Now consider any $r > 0$ and choose $k \in \mathbb{N}$ such that $2^k \leq r \leq 2^{k+1}$. Then

$$Y_r = \frac{\log(Z_r)}{\log(r)} \leq \frac{\log(Z_{2^{k+1}})}{\log(2^k)} = \frac{\log(Z_{2^{k+1}})}{\log(2^{k+1})} \cdot \frac{\log(2^{k+1})}{\log(2^k)} = Y_{2^{k+1}} \cdot \frac{k+1}{k}$$

and

$$Y_r = \frac{\log(Z_r)}{\log(r)} \geq \frac{\log(Z_{2^k})}{\log(2^{k+1})} = \frac{\log(Z_{2^k})}{\log(2^k)} \cdot \frac{\log(2^k)}{\log(2^{k+1})} = Y_{2^k} \cdot \frac{k}{k+1},$$

so

$$\limsup_{r\to\infty} Y_r = \limsup_{k\to\infty} Y_{2^k} \tag{6.13}$$

and

$$\liminf_{r\to\infty} Y_r = \liminf_{k\to\infty} Y_{2^k}. \tag{6.14}$$

Evaluating (6.13) in (6.12) yields the desired statement (6.9).

The proof of (6.10) is almost the same. By the arguments used in (6.11) - (6.12) we obtain

$$\mathbb{P}\left(Y_{r_k} \leq -\epsilon_k + \alpha \text{ for infinitely many } k\right) = 0$$

and therefore

$$\mathbb{P}\left(\liminf_{k\to\infty}(Y_{r_k}) \geq \alpha\right) = 1. \tag{6.15}$$

Evaluating (6.14) in (6.15) yields the desired statement (6.10).

□

We are ready to prove the main theorems.

Proof of Theorem 6.1.1.
Apply Lemma 6.3.2.(i) to Theorem 6.2.1, with $Z_r = |X_r|$ and $\beta = 4$, to obtain

$$\mathbb{P}_{\mathrm{IIC}}\left(\limsup_{r\to\infty}\left(\log_r |X_r|\right) \leq 4\right) = 1. \tag{6.16}$$

Apply Lemma 6.3.2.(ii) to Theorem 6.3.1, with $Z_r = |X_{r,r}|$ and $\alpha = 4$, to obtain

$$\mathbb{P}_{\mathrm{IIC}}\left(\liminf_{r\to\infty}\left(\log_r |X_{r,r}|\right) \geq 4\right) = 1. \tag{6.17}$$

Because $|X_{r,r}| \leq |X_r|$ for all $r \geq 0$ the theorem now follows from (6.16) and (6.17).

□

Proof of Theorem 6.1.2.
Apply Lemma 6.3.2.(i) and 6.3.2.(ii) to Lemma 6.2.5, with $Z_r = |B_r|$ and $\alpha = \beta = 2$, to obtain

$$\mathbb{P}_{\mathrm{IIC}}\left(\limsup_{r\to\infty}\left(\log_r|B_r|\right) \leq 2\right) = \mathbb{P}_{\mathrm{IIC}}\left(\liminf_{r\to\infty}\left(\log_r|B_r|\right) \geq 2\right) = 1.$$

□

Appendix: Immersing Reed

The *strong immersion number* $i(G)$ of a graph G is the size of the largest clique C that is a *strong immersion* of G, meaning that there is an injective function f from (the vertices of) C to $V(G)$ such that

- the vertices in $f(C)$ are connected in G by edge-disjoint paths and
- for each such path only the endpoints intersect $f(C)$.

It is conjectured that for every graph G, $\chi(G) \leq i(G)$ [81, 1] and it is known [49] that $\chi(G) < 3.54 \cdot i(G) + 4$. Recall that $\delta^*(G)$ denotes the degeneracy of G. Inspired by Reed's conjecture ($\chi(G) \leq \left\lceil \frac{\Delta(G)+\omega(G)+1}{2} \right\rceil$ for every graph G), we show the following weaker result.

Theorem. *For every graph G,*

$$\chi(G) \leq \left\lceil \frac{i(G) + \delta^*(G)}{2} \right\rceil \leq \left\lceil \frac{i(G) + \Delta(G)}{2} \right\rceil.$$

The short proof uses *Kempe chains*. In this appendix we use the following definition of a Kempe chain. Given a proper colouring c of a graph G, a vertex $v \in V(G)$ and given two colours a, b, the *ab−Kempe chain in G containing v* is the maximal connected subset $K \subseteq V(G)$ containing v such that each vertex in K is coloured either a or b. Since the colouring is proper, the colours have to alternate along each path in the Kempe chain. Moreover, note that swapping the colours in the Kempe chain yields another proper colouring of G.

Proof of Theorem
Suppose the theorem is false and let G be a counterexample that minimizes $|V(G)| + |E(G)|$. Then $\chi(G) > r := \left\lceil \frac{i(G)+\delta^*(G)}{2} \right\rceil$ and for all $e \in E(G)$ we have $\chi(G-e) \leq \left\lceil \frac{i(G-e)+\delta^*(G-e)}{2} \right\rceil \leq r$, where the second inequality follows because both $i(G)$ and $\delta^*(G)$ are nonincreasing under edge-removal. In fact we have $\chi(G-e) = r$ for all $e \in E(G)$, because otherwise $\chi(G-e) \leq r-1$ and then we could recolour a vertex incident to e to obtain an r−colouring of G; contradiction.
Let w be a vertex of minimum degree in G. Note that $1 \leq \deg(w) \leq \delta^*(G)$. Consider an edge $uw \in E(G)$. Since $G - uw$ has chromatic number r, it follows that G has a proper vertex-colouring c with $r+1$ colours such that w is the unique vertex with

colour $c(w) = r+1$. In $N(w)$, each colour in $\{1, \ldots, r\}$ must occur, since otherwise we could recolour w with one of the missing colours, yielding the contradiction $\chi(G) \leq r$. It follows that at least $r - (\deg(w) - r) \geq 2r - \delta^*(G)$ neighbours of w have a *unique* colour among $N(w)$. Let $x_1, \ldots, x_B$ denote these neighbours. Suppose there is a pair $x_i x_j$ for which the $c(x_i)c(x_j)$ Kempe chain containing x_i does not contain x_j. Then we can swap colours along the Kempe chain to remove the colour $c(x_i)$ from $N(w)$, enabling us to recolour w to $c(x_i)$, again obtaining $\chi(G) \leq r$; contradiction. It follows that for each pair $x_i x_j$ there is a $c(x_i)c(x_j)$ Kempe chain containing x_i and x_j, and thus there is a path from x_i to x_j with alternating colours $c(x_i), c(x_j)$. Note that this path cannot intersect w or any vertex of $\{x_1, x_2, \ldots, x_B\} \setminus \{x_i, x_j\}$, since those vertices are not coloured $c(x_i)$ or $c(x_j)$. Furthermore, note that these paths must be edge-disjoint. It follows that K_{B+1} is a strong immersion of G and so G has a strong immersion of a clique of size $B + 1 = 2r - \deg(w) + 1 \geq 2r - \delta^*(G) + 1 > i(G)$, contradicting that G is a counterexample.

□

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Samenvatting

Dit proefschrift gaat hoofdzakelijk over het efficiënt kleuren van grafen, maar behandelt ook een probleem uit de percolatietheorie. Ieder hoofdstuk bevat de inhoud van een artikel en kan afzonderlijk gelezen worden. Deze samenvatting is bedoeld voor de wiskundig onderlegde leek. Voor meer algemene achtergrond en een uitgebreider overzicht van de verkregen resultaten verwijs ik u graag naar de introductie (Hoofdstuk 1). Voor nog meer details kunt u de abstracts en introducties aan het begin van ieder hoofdstuk lezen.

Grafen zijn formele representaties van netwerken, bestaande uit punten en lijnen die paren punten verbinden. Een punt wordt ook wel een *knoop* genoemd en een verbinding tussen knopen heet ook wel een *tak*. Een voorbeeld is de vriendengraaf, die een knoop heeft voor ieder mens en een tak heeft tussen twee knopen dan en slechts dan als de corresponderende mensen bevriend zijn. We nemen hier optimistisch aan dat 'vrienden zijn' een symmetrische relatie.

We kunnen kleuren toekennen aan de knopen van een graaf. Zo'n kleuring noemen we *correct* als iedere twee takverbonden knopen verschillende kleuren hebben. Het *chromatische getal* $\chi(G)$ van een graaf G is het minste aantal kleuren dat nodig is om de graaf correct te kleuren.

In dit proefschrift bekijken we verschillende canonieke verzamelingen van grafen. Binnen zo'n verzameling identificeren we de (vermoedelijke) graaf of grafen met het hoogst mogelijke chromatische getal en vervolgens bepalen of benaderen we de waarde van dat getal.

Het chromatische getal van een graaf G is altijd *ten hoogste* $\Delta(G)+1$, waarbij $\Delta(G)$ de maximale graad van G is, het maximale aantal buurknopen dat een knoop in G kan hebben. Er bestaat ook een natuurlijke ondergrens: het chromatische getal is altijd *ten minste* het kliekgetal $\omega(G)$ van G, hetgeen gedefinieerd is als het grootste aantal knopen dat paarsgewijs verbonden is door een tak. Samengevat hebben we dus dat $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ voor alle grafen.

Gegeven een graaf H kan men geïnteresseerd zijn in de klasse van grafen van maximale graad ten hoogste Δ die H niet als deelgraaf hebben. Zelfs als we voor H een simpele graaf als de driehoek nemen (drie knopen en drie takken), dan nog is het geen eenvoudige opgave om het chromatische getal van de corresponderende klasse te begrenzen. In de jaren negentig is bewezen dat het chromatische getal voor grafen G in deze klasse ten hoogste $C \cdot \Delta(G)/\log(\Delta(G))$ is, voor een zekere constante C. De best mogelijke bovengrenzen in termen van $\Delta(G)$ zijn niet bekend.

Het kliekgetal is een lokale eigenschap van een graaf en daarom vaak makkelijker te bepalen dan het chromatische getal. Gegeven een klasse grafen $\mathcal{G}$ zijn we daarom blij als er een nietdalende functie f bestaat zodanig dat $\chi(G) \leq f(\omega(G))$ voor alle grafen G in $\mathcal{G}$. Als dit zo is dan heet de klasse $\chi-$begrensd.

Laat G een graaf en laat a en b knopen zodanig dat ac en cb takken zijn, voor een zekere derde knoop c. Als men voor elk zulk paar knopen $\{a, b\}$ de tak ab toevoegt aan G verkrijgen we een nieuwe graaf die het *kwadraat* van G heet. Het kwadraat van G wordt ook wel geschreven als G^2.

De *lijngraaf* $L(G)$ van G is de volgende graaf. De knopen van $L(G)$ zijn de takken van G en de takken van $L(G)$ zijn de *paren* takken in G die een knoop gemeen hebben.

Een klassiek resultaat is dat $\Delta(G) \leq \chi(L(G)) \leq \Delta(G) + 1$ voor alle grafen G. Het chromatische getal van lijngrafen is dus zeer gedetailleerd bekend. Het kwadraat van de lijngraaf is echter veel minder goed begrepen. Het volgende vermoeden van Erdős en Nešetřil uit de jaren 80 speelt een centrale rol in dit proefschrift. Erdős en Nešetřil vermoedden dat $\chi(L(G)^2) \leq \frac{5}{4}\Delta(G)^2$ geldt voor alle grafen G. Als het vermoeden klopt dan is het ook de best mogelijke bovengrens, vanwege zekere grafen genaamd *opgeblazen vijfcykels*. Een eenvoudig argument geeft dat $\chi(L(G)^2) \leq 2\Delta(G)^2$ voor alle G. De beste bekende bovengrens (die geldt voor alle waardes van $\Delta(G)$) geeft echter nauwelijks verbetering, namelijk $\chi(L(G)^2) \leq (2 - \epsilon)\Delta(G)^2$ voor een zeer kleine $\epsilon > 0$.

In hoofdstuk 4 bekijken we de corresponderende vraag voor het kliekgetal. Als het Erdős-Nešetřil vermoeden waar is dan moet zeker ook gelden dat $\omega(L(G)^2) \leq \frac{5}{4}\Delta(G)^2$ voor alle G. In stelling 4.1.5 bewijzen we deze bovengrens voor alle driehoek-vrije grafen. In het zelfde hoofdstuk bewijzen we ook bovengrenzen voor $\omega(L(G)^2)$ onder andere voorwaarden, zoals bijvoorbeeld de afwezigheid van zekere lange paden of cykels als deelgraaf van G. In die gevallen blijken de extremale waardes van $\omega(L(G)^2)$ en $\chi(L(G)^2)$ wel duidelijk verschillend te zijn. We begrenzen $\omega(L(G)^2)$ ook in termen van het zogenaamde Hadwigergetal van G, een niet-lokale parameter die als het ware de grootste 'opgeblazen kliek' van G meet.

In hoofdstuk 3 bekijken we een generalisering van het Erdős-Nešetřil vermoeden. Een *klauw* bestaat uit een knoop met drie buren die paarsgewijs niet verbonden zijn door een tak. In een lijngraaf kan geen klauw zitten. Met andere woorden, alle lijngrafen zijn *klauwvrij*. Het omgekeerde is echter niet waar, dus klauwvrije grafen vormen een grotere klasse dan de lijngrafen. We beschouwen het a priori moeilijkere vermoeden dat $\chi(G^2) \leq \frac{5}{4}\omega(G)^2$ voor alle klauwvrije grafen G. We bewijzen (voor kleine waardes van $\omega(G)$) dat dit algemenere vermoeden in feite equivalent is met het originele vermoeden van Erdős en Nešetřil.

In hoofdstuk 5 onderzoeken we *intersectiegrafen* van families *Jordanoppervlakken* en *Jordancurves*. Grofweg kunnen we een Jordancurve definiëren als een gesloten curve in $\mathbb{R}^2$ die zichzelf niet doorsnijdt. Het is een soort vervormde cirkel. Een Jordanoppervlak is dan grofweg een opgevulde Jordancurve. Gegeven een familie deelverzamelingen $\mathcal{F} = \{A_1, A_2, \ldots\}$ kunnen we de intersectiegraaf $G(\mathcal{F})$ definiëren als volgt. Iedere knoop van $G(\mathcal{F})$ correspondeert met een verzameling van $\mathcal{F}$ en er is een tak tussen twee gegeven knopen dan en slechts dan als de twee corresponderende verzamelingen niet disjunct zijn. In hoofdstuk 5 beschouwen we de intersectiegraaf $G(\mathcal{F})$ van een familie $\mathcal{F}$ bestaande uit Jordanoppervlakken waarvan ieder paar ten hoogste 1 gemeenschappelijk punt in $\mathbb{R}^2$ heeft. Voor deze grafen bewijzen we de scherpe bovengrens $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 1$ (indien $\omega(G(\mathcal{F})) \geq 490$). De klasse van deze intersectiegrafen is dus extreem χ-begrensd. We bewijzen vergelijkbare bovengrenzen voor zekere families Jordancurves.

Een ander onderwerp dat in dit proefschrift aan bod komt is een vermoeden van Bollobás, Catlin en Eldridge uit de jaren zestig, hierna het BEC-vermoeden genoemd. Het betreft een generalisering van een stelling over *gelijkkleuringen* (*equitable colourings* in het Engels). Een graaf heet gelijkkleurbaar met k kleuren als de knopen correct gekleurd kunnen worden met k kleuren, zodanig dat de cardinaliteiten van de kleurklasses paarsgewijs ten hoogste 1 verschillen. Dat wil zeggen, ieder van de k kleuren wordt aan ongeveer het zelfde aantal knopen toegekend. Hajnal en Szemerédi bewezen in de jaren zestig een vermoeden van Erdős dat alle grafen G gelijkkleurbaar zijn met $\Delta(G) + 1$ kleuren. Het BEC-vermoeden is een generalisering van deze stelling, in termen van de volgende terminologie. Twee grafen G_1 en G_2 zijn *trouwbaar* (*packable* in het Engels) als G_1 een deelgraaf is van het complement van G_2. Men kan veel graaftheoretische vragen herformuleren in de taal van trouwbaarheid. Bijvoorbeeld, H is een deelgraaf van G dan en slechts dan als H en het complement van G trouwbaar zijn. Het BEC-vermoeden stelt dat twee grafen G_1 en G_2 met n knopen trouwbaar zijn als $(\Delta(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$. In hoofdstuk 2 bewijzen we dit vermoeden onder de extra voorwaarde dat G_1 geen viercykel als deelgraaf heeft en $\Delta(G_1) > 34 \cdot \Delta(G_2)$. Algemener bewijzen we dit onder de exclusie van zekere complete bipartiete grafen. Als bijresultaat verkrijgen we dat zekere klasses van grafen gelijkkleurbaar zijn met $\Delta(G)$ kleuren (in plaats van de $\Delta(G) + 1$ kleuren gegarandeerd door de Hajnal-Szemerédistelling; een verbetering die niet zo marginaal is als het wellicht klinkt).

In hoofdstuk 6 behandelen we ten slotte een probleem uit de percolatietheorie dat niets van doen heeft met graafkleuringen. Men beschouwt een getal $p \in [0, 1]$ en een graaf G met takken $E(G)$. Vervolgens verkrijgt men een random deelgraaf van G door iedere tak van $E(G)$ te behouden met kans p en te verwijderen met kans $1 - p$, onafhankelijk van elkaar. Een *cluster* in deze random graaf is een maximale verzameling knopen zodanig dat tussen iedere twee van die knopen er een pad is in de random graaf. Het cluster heet oneindig als het aantal knopen erin niet eindig is. Veel onderzoek is gedaan naar de graaf G met knopenverzameling $\mathbb{Z}^d$ en met een tak tussen twee punten dan en slechts dan als hun onderlinge (Euclidische) afstand 1 is. Men kan afleiden dat er een *kritieke parameter* $p_c \in [0, 1]$ bestaat zodanig dat met kans 1 de random deelgraaf van G geen oneindig cluster bevat als $p < p_c$ en één uniek oneindig cluster bevat als $p > p_c$. Wat er gebeurt in $p = p_c$ is een belangrijke open vraag voor dimensies $3 \leq d \leq 10$. In dimensie $d = 2$ en in 'hoge dimensies' $d \geq 11$ is wel veel bekend. We weten dan dat er geen oneindig cluster is in p_c, met kans 1. We beschouwen vanaf nu het geval $d \geq 11$. Het blijkt dat men dan een kansmaat $\mathbb{P}_{\mathrm{IIC}}$ kan definiëren die als het ware conditioneert op de gebeurtenis dat er *toch* een oneindig cluster is in p_c. De afkorting IIC staat voor Incipient Infinite Cluster, wat zoveel wil zeggen als 'het zich vormende oneindige cluster'. In hoofdstuk 6 bewijzen we dat het oneindige cluster 4−dimensionaal is, in de zin dat elke bal van (voldoende grote) straal r rondom de oorsprong ongeveer r^4 punten van het oneindige cluster bevat, met $\mathbb{P}_{\mathrm{IIC}}$−kans 1. Ter vergelijking: voor $p < p_c$ is het oneindige cluster nonexistent en dus 0−dimensionaal. Voor $p > p_c$ heeft het oneindige cluster plotseling de zelfde dimensie als de hele omliggende graaf $\mathbb{Z}^d$, namelijk d.

Acknowledgement

First of all, I would like to thank my supervisors Eric and Ross. They guided me but also let me free to think about any topic that interested me. I am grateful for collaboration with and/or visits to Rob van den Berg, Louis Esperet, Markus Heydenreich, Gwenaël Joret, Tobias Müller, François Pirot and Jean-Sébastien Sereni. Furthermore, I thank the members of the manuscript committee for their efforts and the time they spent on my work.

In Nijmegen, I enjoyed conversations with Henk Don, about mathematics and old fashioned Dutch proverbs. I would like to thank my (former) colleagues Frank Roumen, Julius Witte and Ruben Stienstra for organizing board game afternoons. They made me feel at home in Nijmegen. In a later stage, Stijn Cambie, Rémi de Joannis de Verclos and Kamyar Saeedi were valued colleagues and roommates.

My parents have supported me at all stages of my education and personal life, from elementary school to university. Life as a researcher is not always bright; it can come with many disappointments. During such times I could always rely on my parents. Their support was very important to me. In this context, I would also like to warmly thank Xiaochen Zheng as well as Jesse Hemerik, my brother Diederik and ... Ross. Ross is not just a supervisor, he is also a very good friend!

Curriculum Vitae

Wouter Cames van Batenburg was born on December $10, 1988$ in Den Haag, The Netherlands. From 2001 to 2007 he attended the gymnasium Haganum in Den Haag. He started studying mathematics and physics at Leiden University in 2007, and obtained both B.Sc. degrees in 2011. He continued with the M.Sc. program in mathematics at Leiden University. After graduating with the distinction *cum laude* in December 2013, he started as a Ph.D. student at Radboud University, supervised by Prof. Eric Cator and Dr. Ross Kang. The Ph.D. project initially involved problems in the fields of Last Passage Percolation and interacting particle systems, but later evolved more in the direction of extremal graph theory and graph colouring. In April 2018 he started as a postdoctoral researcher at the Université Libre de Bruxelles in the group of Dr. Gwenaël Joret.

Figure 6.1: Picture by Xiaochen Zheng.