Optimized density argument for Erdős 1033.

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Rickyc and chatgpt at erdosproblems.com gave an improved upper bound for Erdős problem number 1033, via a suitable blow-up of the 5-vertex butterfly graph: two triangles sharing a vertex. That resulted in an 1.46393 upper bound which improved upon the previous best known $2(\sqrt{3}-1) \approx 1.46410$. This document contains (a proof sketch of) an optimization of the density parameters, using the same type of blow-up, leading to $1.463878\ldots$, a root of some polynomial of degree 7. An analysis of blow-ups of several other graphs (among which all graphs on ≤ 8 vertices), did not yet produce any better bound.

Lemma 1. *For every N large enough, there exists a graph on N vertices and $> \frac{N^2}{4}$ edges, such that every triangle has degrees summing to $\leq 1.463878 \cdot N$.*

Proof. Proof sketch. Consider the graph on five vertices a, b, c, d, e formed by two triangles abc and ade sharing one vertex a. Let

$$(x_a, x_b, x_c, x_d, x_e) = (0.42727843, 0.24709063, 0.07094347, 0.12734373, 0.12734373).$$

Fix a sufficiently large integer N. Blow up vertices a, b, c, d, e to independent sets of sizes (integers closest to) $x_aN, x_bN, x_cN, x_dN, x_eN$. With slight abuse of notation, we call these independent sets a, b, c, d, e as well. Make $\{ab, bc, ad, ae\}$ complete bipartite graphs, and make ac bipartite with $f_{ac} \cdot x_a x_c N^2$ edges, where $f_{ac} \approx 0.43512057$. For N large enough, we can ensure that all vertices in a given part of the bipartition have the same number of neighbours in the other part, up to lower order $o(N)$ terms, for instance by starting with the complete bipartite graph between a and c and taking a random subgraph by independently keeping each edge with probability f_{ac}. Similarly, make de bipartite with $f_{de}\, x_d x_e N^2$ edges, where $f_{de} \approx 0.30104858$, again ensuring the degrees are asymptotically evenly distributed.

Let G_N denote the resulting graph.

As N goes to infinity, the density $\frac{|E(G_N)|}{N^2}$ converges to $x_a x_b + x_b x_c + f_{ac} x_a x_c + x_a x_d + x_a x_e + f_{de} x_d x_e \approx \frac{1}{4}$. The normalized vertex degrees (i.e. the degree of a vertex divided by N) are

$$\begin{aligned}
\deg_a &= x_b + f_{ac}\, x_c + x_d + x_e \approx 0.53264706, \\
\deg_b &= x_a + x_c \approx 0.49822191, \\
\deg_c &= x_b + f_{ac}\, x_a \approx 0.43300826, \\
\deg_d &= \deg_e = x_a + f_{de}\, x_d \approx 0.46561508.
\end{aligned}$$

All possible triangles are of the form abc or ade, and hence have the same normalized degree sum:

$$\deg_a + \deg_b + \deg_c = \deg_a + \deg_d + \deg_e \approx 1.463877.$$

A sufficiently small increase in one of the partial densities makes the edge density strictly greater than $1/4$, while keeping every triangle degree-sum below $1.463878\,N$. $\square$

Departing from the same 5-vertex graph and argument as in Lemma 1, writing out the density equations and optimizing, in particular setting equal the degree-sums of triangles of both types abc and ade, yields the following exact form, parameters computed by ChatGPT:

Lemma 2. *Let $y_0 \approx 1.463877226961\ldots$ denote the smallest positive real root of the polynomial $P(y) = 384y^7 + 1744y^6 - 832y^5 - 86712y^4 + 379016y^3 - 697963y^2 + 617212y - 215708$.*

Then for every $y^+ > y_0$ and sufficiently large integer N, there exists a graph on N vertices and $> \frac{N^2}{4}$ edges, such that every triangle has degrees summing to $\leq y^+ \cdot N$.

The exact density parameters of the blown-up graph are as follows. Define

$$\nu = \frac{76y_0^3 - 388y_0^2 + 703y_0 - 436}{4y_0^3 - 428y_0^2 + 1269y_0 - 953}$$

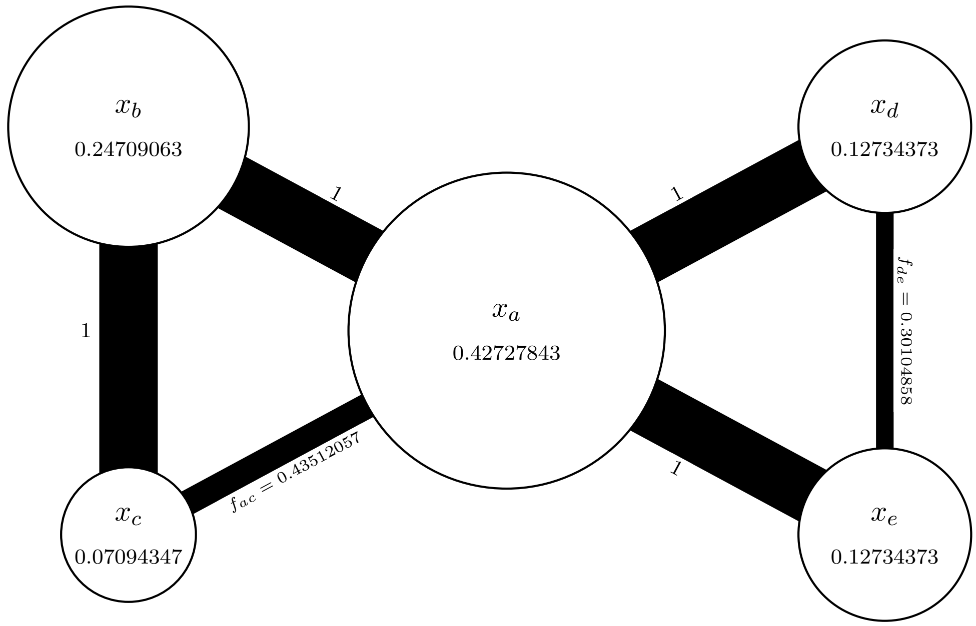
$$\alpha = \frac{-13988y_0^3 + 69464y_0^2 - 131761y_0 + 87992}{25500y_0^3 - 203232y_0^2 + 389219y_0 - 214046}.$$

$$\lambda = \frac{\nu - (\nu + 2)\alpha}{2\nu(\nu - 1)^2}.$$

Then

$$\begin{aligned}
x_a &= \alpha, \\
x_b &= \frac{\lambda}{2}\left(4\nu^2 - 5\nu + 2\right), \\
x_c &= \frac{\lambda\nu}{2}, \\
x_d = x_e &= \frac{1 - x_a - x_b - x_c}{2}, \\
f_{ac} &= 1 - \frac{2\lambda\nu(\nu - 1)}{\alpha}, \\
f_{de} &= \frac{\lambda}{2x_d}.
\end{aligned}$$

Despite the weird asymmetries, the optimization arguments yielded that these values for the densities form the *unique* minimizer for blow-ups of the butterfly graph (up to isomorphism).


x_b
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x_d
0.12734373
1
1
1
x_a
0.42727843
$f_{de} = 0.3010485 8$
x_c
0.07094347
$f_{ac} = 0.43512057$
1
x_e
0.12734373
1