Domination–packing ratio below 10 for unit disk graphs

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Abstract

For every finite unit disk graph, the domination number is at most $\frac{18\sqrt{3}}{\pi} \approx 9.924$ times the packing number. This improves the bound of 16 due to Gutiérrez and Paul, who in turn improved the bound of 32 due to Bonamy, Csikós, Gujgiczer and Yuditsky. The best known lower bound on the optimal constant remains 3.

1 Introduction

Duality between packing and covering is a well-studied theme in structural graph theory: given a family of 'nice' objects, one seeks either many pairwise disjoint nice objects, or a small set that intersects every nice object. Statements of this type fall under the umbrella of *Erdős–Pósa* theorems. Most Erdős-Pósa theorems can be encoded in terms of some class $\mathcal{H}$ of hypergraphs. For every hypergraph $H \in \mathcal{H}$, the nice objects correspond to the hyperedges of H. If ρ denotes the maximum number of pairwise disjoint hyperedges and γ the minimum size of a vertex set that nontrivially intersects every hyperedge of H, then a *bounding function* for $\mathcal{H}$ (if it exists!) is a function f such that $\gamma \leq f(\rho)$ for every $H \in \mathcal{H}$. In the foundational result of Erdős and Pósa [EP65], the hyperedges are the cycles of some fixed graph: they proved that every graph either contains k vertex-disjoint cycles, or a set of $\mathcal{O}(k \log k)$ vertices meeting every cycle. Thus, for cycles there exists a bounding function $f(\rho) = O(\rho \log \rho)$. Since then, the same question has been studied for many more structured objects, such as directed cycles [RRST96], matroid circuits [GK09], cycles in group-labelled graphs [GHK+25] and graph minor models [RS86, CHJR19], see also the survey [RT17].

A particularly relevant special case for this note is that of *ball hypergraphs*. Given a graph G, a ball hypergraph of G has vertex set $V(G)$ and hyperedges given by graph balls $B_r(v) = \{u \in V(G) : \mathrm{dist}_G(u, v) \leq r\}$, possibly with different radii. In [BCE+21], Bousquet et al. proved that for every graph Γ there exists a constant c such that every ball hypergraph of a Γ-minor-free graph satisfies $\gamma \leq c \cdot \rho$. In particular this gave a linear bounding function for ball hypergraphs of planar graphs, even when not all balls have the same radii, confirming a conjecture of Chepoi, Estellon and Vaxès [CEV07]. Earlier progress, with constants depending on the radius or on weak colouring numbers, was obtained by Dvořák [Dvo13, Dvo19].

In this note we focus on the very special case that all balls have radius 1. Then the covering problem reduces to the usual domination problem. More precisely, from now on, for a graph G, the *domination number* $\gamma(G)$ is the minimum size of a set $D \subseteq V(G)$ such that $N_G[D] = V(G)$, equivalently, such that D has non-empty intersection with every radius-1 ball. And the *packing number* $\rho(G)$ is the maximum size of a set $P \subseteq V(G)$ whose closed neighborhoods are pairwise disjoint, equivalently, whose vertices are pairwise at distance larger than 2. Because the term

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'packing number' is heavily overloaded in graph theory, referring to numerous very different notions of disjoint structures, a common alternative name for $\rho(G)$ is the 2-*independence number*.

Clearly $\rho(G) \leq \gamma(G)$. This inequality is sharp for trees [MM75], but it can fail arbitrarily badly even in sparse graphs: Dvořák [Dvo13] constructed bipartite 3-degenerate graphs with 2-independence number 2 and unbounded domination number. An elegant basic positive bound is $\gamma(G) \leq \Delta(G)\rho(G)$ [HLR11], where $\Delta(G)$ denotes the maximum degree of G.

Recently, Bonamy, Csikós, Gujgiczer and Yuditsky [BCGY25] initiated a systematic study of graph classes with constant domination-packing ratio. Among many other results, they proved that $\gamma(G) \leq 10\rho(G)$ for planar graphs and $\gamma(G) \leq 32\rho(G)$ for unit disk graphs. A unit disk graph is a graph admitting a representation $p : V(G) \to \mathbb{R}^2$ such that, for distinct vertices u, v, we have $uv \in E(G)$ if and only if $\|p(u) - p(v)\| \leq 1$. A second recent work of Bonamy, Dvořák, Michel and Mikšaník [BDMM26] gives an exact characterization of the monotone graph classes of bounded average degree that admit a linear bounding function $\gamma(G) \leq c \cdot \rho(G)$, for some constant c that only depends on the class.

The constants above have since been improved in the two classes closest to this note. Dúcz and Gujgiczer [DG26] improved the planar bound from 10 to 7, while the best known lower bound for planar graphs is 3. For unit disk graphs, Gutiérrez and Paul [GP26] improved the bound from 32 to 16 and also gave an infinite family with ratio 3. In this note we describe a modest further improvement of the upper bound for unit disk graphs.

Theorem 1.1. *For every finite unit disk graph G,*

$$\gamma(G) \leq \frac{18\sqrt{3}}{\pi}\,\rho(G) < 10\,\rho(G).$$

The proof takes a maximal independent (and hence dominating) set I of the unit disk graph, considers the points $X \subseteq \mathbb{R}^2$ that represent I, and then uses a randomly shifted triangular grid to extract a large subset $Y \subseteq X$ of points that are at pairwise Euclidean distance > 2. The vertices corresponding to Y have disjoint closed neighborhoods and thus form a packing

2 The proof

A subset $P \subset \mathbb{R}^2$ is called *c-separated* if $d(x, y) > c$ for all distinct $x, y \in P$. Here $d(x, y)$ denotes the Euclidean distance between x and y. To prove Theorem 1.1, we first derive an auxiliary lemma that finds a large 2-separated set inside a given 1-separated set.

Lemma 2.1. *Let $X \subset \mathbb{R}^2$ be a finite 1-separated point set. Then there exists a 2-separated subset $Y \subseteq X$ of size $|Y| \geq \frac{\pi}{18\sqrt{3}}\,|X|$.*

Proof. Let Λ be the triangular lattice with shortest nonzero vector of length 3. For instance, take $\Lambda = \left\{ m(3,0) + n\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right) : m, n \in \mathbb{Z} \right\}$. A fundamental parallelogram of Λ has area $\det(\Lambda) = \frac{9\sqrt{3}}{2}$. Let

$$S = \bigcup_{\lambda \in \Lambda} B(\lambda, 1/2),$$

where $B(\lambda, 1/2)$ denotes the open Euclidean disk of radius $1/2$ centred at λ. For a translation vector $t \in \mathbb{R}^2$, define

$$Y_t = X \cap (S + t).$$

We first observe that Y_t is always 2-separated. Indeed, suppose $x, x' \in Y_t$ are distinct. If x and x' lie in the same open disk of radius $1/2$, then $d(x, x') < 1$, contradicting the hypothesis on

X. Hence they lie in two distinct translated disks. The centres of distinct disks in $S+t$ are at distance at least 3. Since the disks are open and have radius $1/2$, we get $d(x,x') > 3 - \frac{1}{2} - \frac{1}{2} = 2$. Thus Y_t is 2-separated.

Now choose t uniformly at random from a fundamental parallelogram of Λ. For every fixed point $x \in X$, the probability that $x \in S+t$ is exactly the density of S, namely

$$\frac{\text{area}(B(0,1/2))}{\det(\Lambda)} = \frac{\pi/4}{9\sqrt{3}/2} = \frac{\pi}{18\sqrt{3}}.$$

Therefore the expected number of points in Y_t is $\mathbb{E}|Y_t| = \sum_{x\in X} \mathbb{P}(x \in S+t) = \frac{\pi}{18\sqrt{3}}|X|$. So for at least one translation t, we have $|Y_t| \geq \frac{\pi}{18\sqrt{3}}|X|$. □

Proof of Theorem 1.1. Fix a unit disk representation $p : V(G) \to \mathbb{R}^2$ such that, for distinct vertices $u, v \in V(G)$,

$$uv \in E(G) \iff d(p(u), p(v)) \leq 1.$$

Let I be a maximal independent set of G. Due to maximality I is dominating, so $\gamma(G) \leq |I|$. Since I is independent in G, we have $d(p(u), p(v)) > 1$ for all distinct $u, v \in I$. Applying Lemma 2.1 to the point set $p(I)$, we obtain a 2-separated subset $Y \subseteq p(I)$ with $|Y| \geq \frac{\pi}{18\sqrt{3}}|I|$.

Let $P = \{u \in I : p(u) \in Y\} \subseteq I$ be the corresponding set of vertices of G. We claim that P is a packing in G. Indeed, if distinct vertices $u, v \in P$ had a common vertex in their closed neighbourhoods, say $w \in N_G[u] \cap N_G[v]$, then due to the triangle inequality $d(p(u), p(v)) \leq d(p(u), p(w)) + d(p(w), p(v)) \leq 2$, contradicting that Y is 2-separated. Thus P is a packing, and so $\rho(G) \geq |P| \geq \frac{\pi}{18\sqrt{3}}|I| \geq \frac{\pi}{18\sqrt{3}}\gamma(G)$. □

3 Further improvements?

Any improvement of the constant $c = \frac{\pi}{18\sqrt{3}}$ in Lemma 2.1 directly implies an improvement $\gamma(G) \leq \frac{1}{c}\rho(G)$ in Theorem 1.1. Our gut feeling says that $c \approx \frac{1}{6}$ is achievable. However, we suspect the "hail shot" approach of choosing some fixed lattice independent of P, and then taking a random translation, has almost been optimized. Computations indicate that small improvements are possible by slightly adapting the shape and size of the radius $1/2$ balls around the lattice points, or by taking a random rotation in addition to a random translation. However, these modifications make the proofs much more technical and uggly, in return for only minor improvements. So far, the best ratio that we could obtain in this way is $\gamma(G) \leq \frac{9\pi}{9\sqrt{3}-4\pi}\rho(G) \approx 9.36\,\rho(G)$. For significant further improvements, a more direct approach tailored to the specific structure of the given point sets P and the Euclidean metric may be necessary.

Availability. This note is intended as a brief communication and will be made available on the author's GitHub page. As of today (July 9, 2026) there are no plans to submit it to arXiv or a journal.

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