The space of greedy list-colourings

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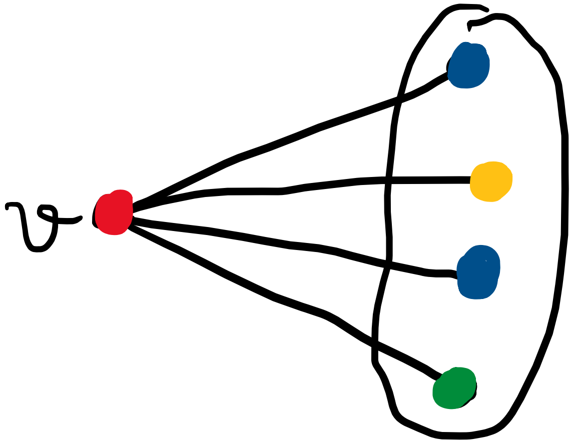
Based on joint works with Stijn Cambie, Daniel Cranston, Ewan Davies, Jan van den Heuvel and Ross Kang.

Ottawa, CANADAM 2025, May 2025.

Basic greedy bound

Graph G, maximum degree Δ, chromatic number χ

$$\boxed{\chi \leq \Delta + 1}$$



$|N(v)| \leq \Delta$

Proof

Induction: $\exists$ $(\Delta+1)$ – colouring c of $G - v$.

Then colour v greedily from $[\Delta+1] \setminus c(N(v)) \neq \phi$

$\square$

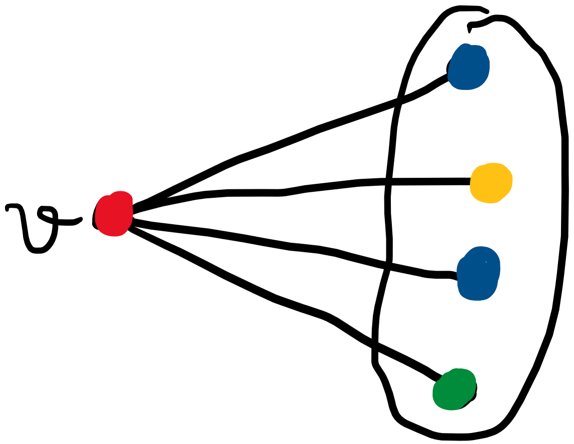
This also works for **list-colouring**.

$L: V(G) \to 2^{\mathbb{N}}$ is a **list-assignment**.

An **L-colouring** is a proper colouring

$c: V(G) \to \mathbb{N}$ s.t. $c(v) \in L(v)$, $\forall v \in V(G)$.

By same argument ...



$|N(v)| \leq \Delta$

$\exists$ L-colouring of $\mathcal{G}$ if

$$\boxed{|L(v)| \geq \Delta + 1 \qquad \forall\, v \in V(\mathcal{G}).}$$

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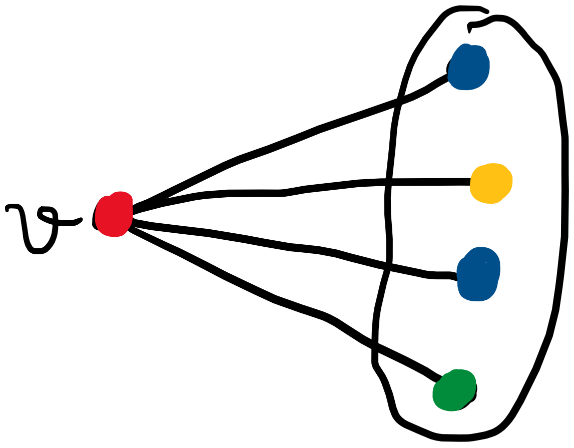

$|N(v)| = \deg(v) \leq \Delta$

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$$\boxed{|L(v)| \geq \Delta + 1 \qquad \forall\, v \in V(\mathcal{G}).}$$

& also if

$$\boxed{|L(v)| \geq \deg(v) + 1 \qquad \forall\, v \in V(\mathcal{G}).}$$

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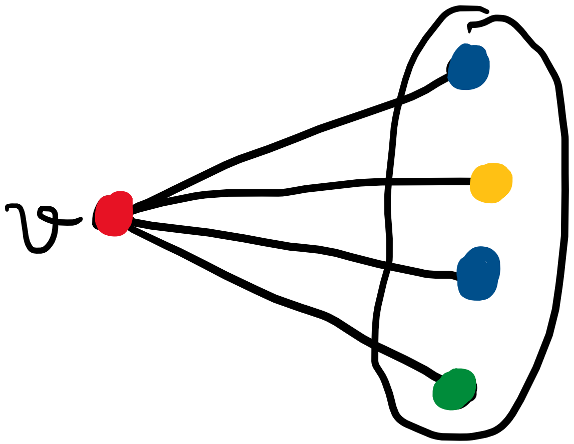

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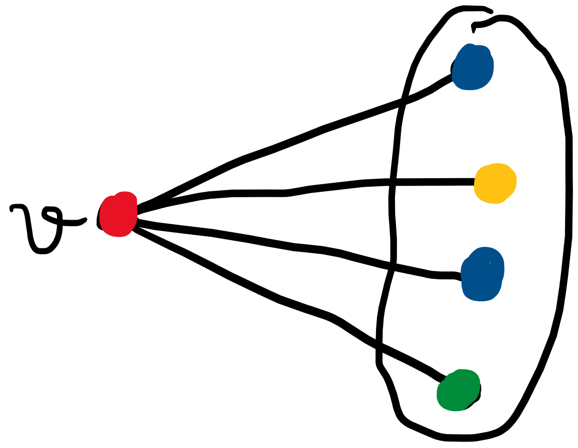

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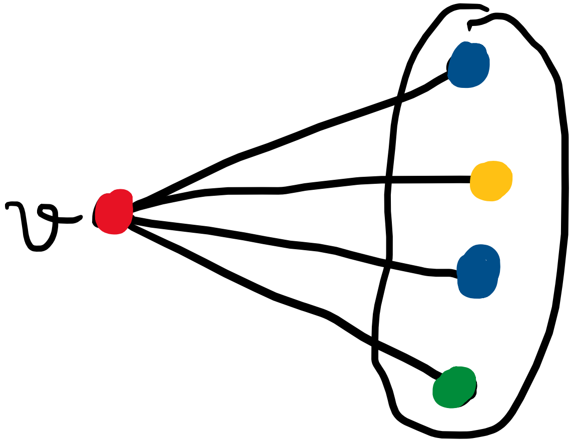
$|L(v)| \ge \deg(v) + 1 \quad \forall v \in V(\mathcal{G})$.

Choose $c(v)$ from non-empty $L(v) \setminus c(N(v))$.

By same argument...



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deg + 1 assignment

So far obtained <u>existence</u> of $\geq$ one L-colouring.

But ...

want to understand entire space of L-colourings.

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But ...

want to understand entire space of L-colourings.

1. count # L-colourings?
2. flexible/ balanced everywhere?
3. How similar/close are the L-colourings to each other?

Count # L-colourings

Recall: $|L(v)| \geq \deg(v)+1 \quad \forall v \in V(\mathcal{G})$.

<u>Observation</u>: $\mathcal{G}$ connected $\implies \exists$ exponentially many L-colourings

Count # L-colourings

Recall : $|L(v)| \geq \deg(v)+1 \quad \forall v \in V(\mathcal{G})$.

<u>Observation</u> : $\mathcal{G}$ connected $\Rightarrow \exists \geq 2^{n-1}$ L-colourings
n vertices

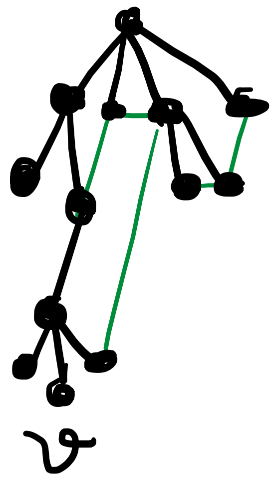
<u>Proof</u>

Count # L-colourings

Recall: $|L(v)| \geq \deg(v)+1 \quad \forall v \in V(G)$.

Observation: G connected $\Rightarrow \exists \geq 2^{n-1}$ L-colourings
n vertices

Proof Choose spanning tree T.
Choose leaf v of T.
$\exists \geq \deg(v)+1 \geq 2$ choices for colour $c(v)$.
Apply induction to $G-v$, removing $c(v)$ from the lists of v's neighbours. $\Rightarrow 2^{n-2}$ L-colourings of $G-v$ $\square$



So $\exists$ many L-colourings.

Are they also „flexible / balanced / tweakable" everywhere?

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Are they also „flexible / balanced / tweakable" everywhere?

yes

Lemma (Cambie, CvB, Davies, Kang, 2023)

$\exists$ probability distribution on L-colourings c s.t.

$$\mathbb{P}(c(v) = x) = \frac{1}{|L(v)|}$$

$\forall v \in V(G) \quad \forall x \in L(v)$.

"(G, L) admits a fractional list packing"

At each vertex, every colour is equally likely.

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- True if $|L(v)| \geq \deg(v)+1$ for all v.
- False $|L(w)| = \deg(w)$ for just one w.
- True $|L(v)| \geq$ pathwidth $+1$
- unknown $|L(v)| \geq$ treewidth $+1$.
- False $|L(v)| \geq$ degeneracy $+1$.

$\exists$ probability distribution on L-colourings c s.t.

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$\forall v \in V(G) \quad \forall x \in L(v)$.

„(G, L) admits a fractional list packing"

Problem Characterize (G, L) with

$$|L(w)| = \deg(w) \quad \text{for some } w$$

$$|L(v)| = \deg(v)+1 \quad \text{for all other } v$$

that do not admit a fractional list packing.

(e.g. cliques)

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$\forall v \in V(G) \quad \forall x \in L(v)$.

3 How similar / close are the L-colourings to each other?

③ How similar / close are the L-colourings to each other?

Def Reconfiguration graph $R_L(G)$ of G has

vertices: $V(R_L) = \{L\text{-colourings of } G\}$.

edges: $c_1 c_2 \in E(R_L) \iff c_1$ & c_2 differ on precisely one vertex

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Q1 $R_L(G)$ connected?

Q2 if so, how small is its diameter?

③ How similar / close are the L-colourings to each other?

Def Reconfiguration graph $R_k(G)$ of G has

vertices: $V(R_k) = \{ k\text{-colourings of } G \}$.

edges: $c_1 c_2 \in E(R_k) \iff c_1 \,\&\, c_2$ differ on precisely one vertex

$k \in \mathbb{N}$

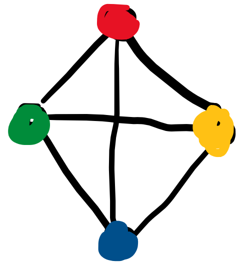
Q1 $R_k(G)$ connected?

Q2 if so, how small is its diameter?

$\underline{Q_1}$ $\longrightarrow$ NOT always connected

if $k \leq \Delta+1$ available colours.

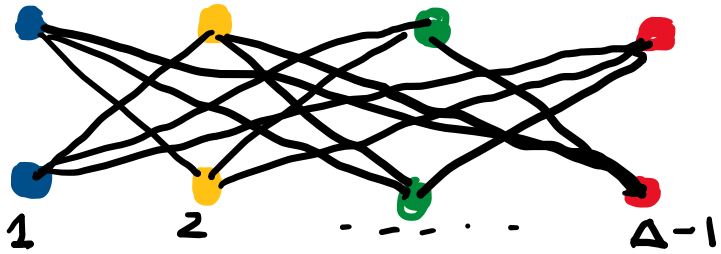
$\underline{\text{Exp.}}$



= frozen 4-colouring of K_4

= isolated vertex of $R_4(K_4)$.

$\underline{\text{Exp.}}$ $G := K_{\Delta,\Delta}$ − Perfect matching.



= frozen $(\Delta+1)$-colouring

= isolated vertex of $R_{\Delta+1}(G)$.

So need $k \geq \Delta + 2$ colours.

<u>Thm</u> (Jerrum, 1995)

$R_{\Delta+2}(G)$ is connected (with diameter $O(2^n)$).

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$R_{\Delta+2}(\mathcal{G})$ is connected (with diameter $\mathcal{O}(2^n)$).

<u>Thm</u> (Cereceda, 2007)

$$\mathrm{diam}(R_{\Delta+2}(\mathcal{G})) = \mathcal{O}(n^2)$$

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Thm (Jerrum, 1995)

$R_{\Delta+2}(G)$ is connected (with diameter $O(2^n)$).

Thm (Cereceda, 2007)

$\operatorname{diam}(R_{\Delta+2}(G)) = O(n^2)$

Thm (Cambie, CvB, Cranston, 2023)

$\operatorname{diam}(R_{\Delta+2}(G)) \leq 2n$

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Thm (Cambie, CvB, Cranston, 2023)

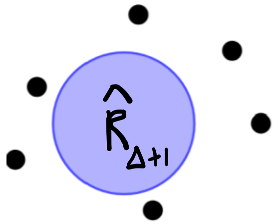
If $|L(v)| \ge \deg(v)+2$ then

$$\operatorname{diam}(R_L(G)) \le 2n.$$

Returning to $\Delta+1$ ... despite frozen colourings ...

Thm (Feghali, Johnson, Paulusma, 2016)

If $\mathcal{G}$ connected, not path or cycle, then $R_{\Delta+1}(\mathcal{G})$ is union of isolated vertices and at most one nontrivial component $\hat{R}_{\Delta+1}(\mathcal{G})$ with diameter $O(n^2)$.



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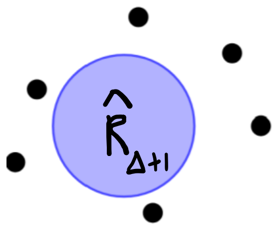


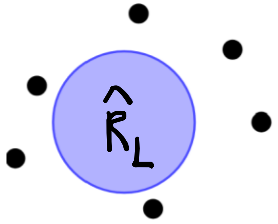
Thm (Bousquet, Feuilloley, Heinrich, Rabie, 2024)

If also min degree ≥ 3 then diameter $O(\Delta^{\Delta} \cdot n)$.

Our work: generalising to local lists

Thm (Cambie, CvB, Cranston, vd Heuvel, Kang, 2025+)

If G connected, not path or cycle, and $\boxed{|L(v)| \geq \deg(v)+1 \quad \forall v}$, then $R_L(G)$ is union of isolated vertices and at most one nontrivial component $\hat{R}_L(G)$ with diameter $O(n^2)$.



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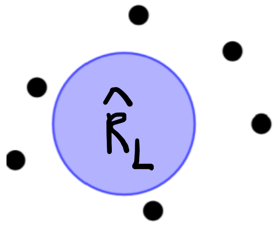
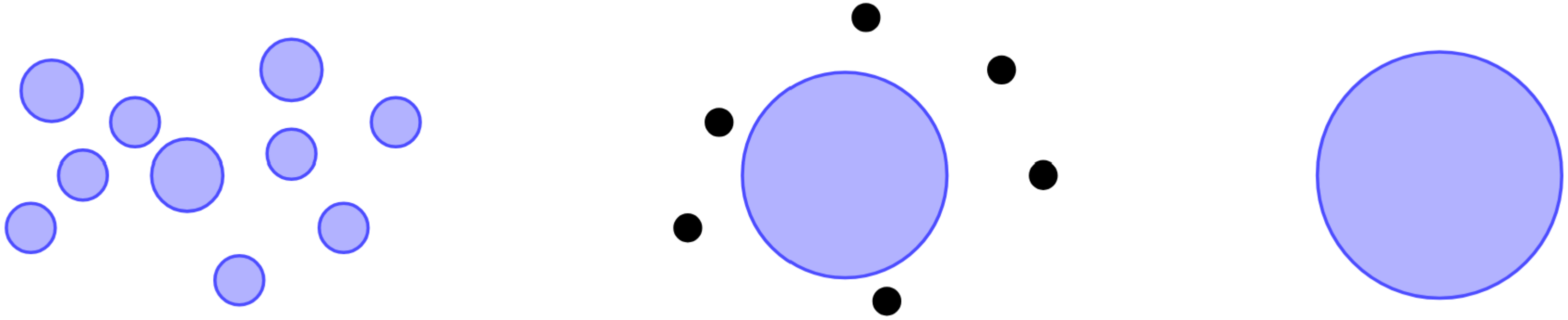


and # isolated vertices is negligible if $\Delta \ll n$.

Sensitive to the local constraints



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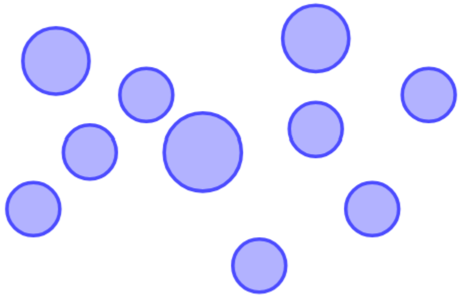
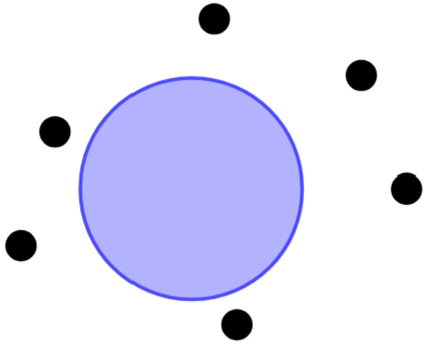


$|L(w)| \le \deg(w)$ for some w

$R_L(G)$ may shatter into many large components.



$|L(v)| = \deg(v)+1$ for all v

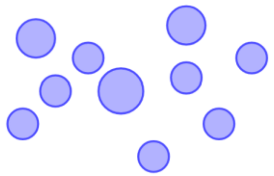
$R_L(G)$ connected up to isolated vertices

$|L(w)| \ge \deg(w)+2$ for some w

$R_L(G)$ connected

Shattering observation

If $|L(w)| = \deg(w)$ for some w
and $|L(v)| \geq \deg(v)+1$ for all other v,
Then $R_L(G)$ could have many large components.



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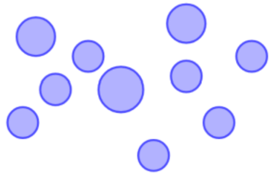
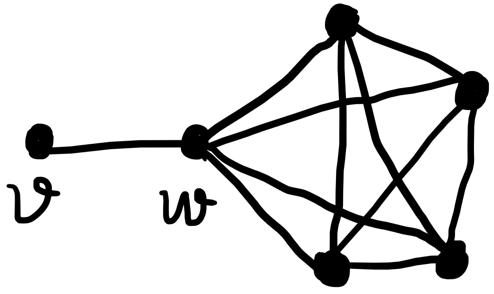


Exp. Take K_n + edge vw,

- Lists $\{1, 2, \dots, n\}$ on K_n
- List $\{n+1, \dots, n+z\}$ on v.



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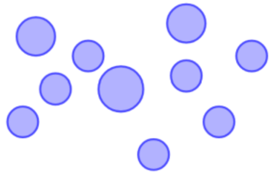

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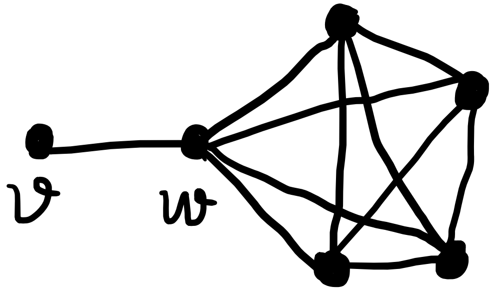

$\Rightarrow$ All L-colourings are frozen on K_n but $\exists$ z choices for colour of v.
$\Rightarrow$ $R_L(G)$ has $n!$ components of size $z \gg 1$ $\square$.

Key Lemma

If $|L(w)| \geq \deg(w)+2$ for some w

and $|L(v)| \geq \deg(v)+1$ for all other v,

then $\operatorname{diam}(R_L(G)) \leq \left(\frac{3}{2} + o(1)\right) n^2$.

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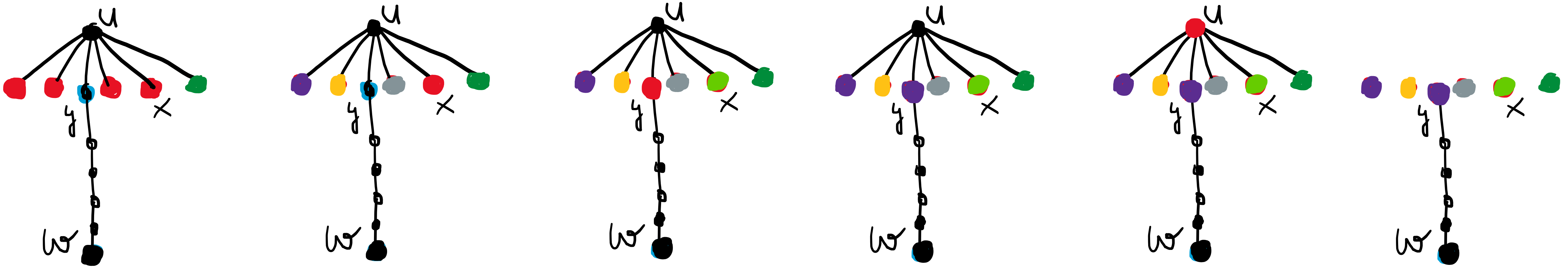
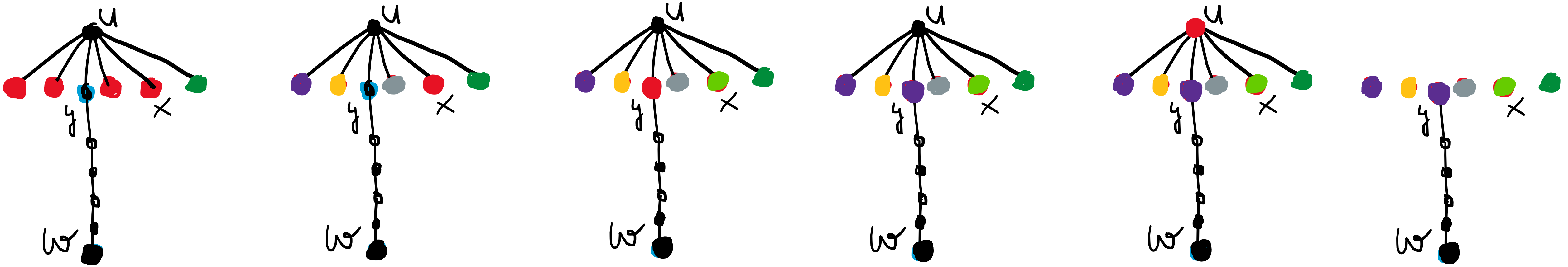
Proof sketch: Choose $u \neq w$. Goal recolour u to



Proof sketch of Key Lemma

Goal recolour u to

Can always recolour along shortest w, u path P



many neighbours with colour

x unique neighbour of u with colour

y unique neighbour of u with colour

NO neigbour with colour

recolour u to

Induction on G - u.

(& Remove from lists of neighbours of u)

$\leq \deg(u)$ steps

$\lesssim \#P$ steps

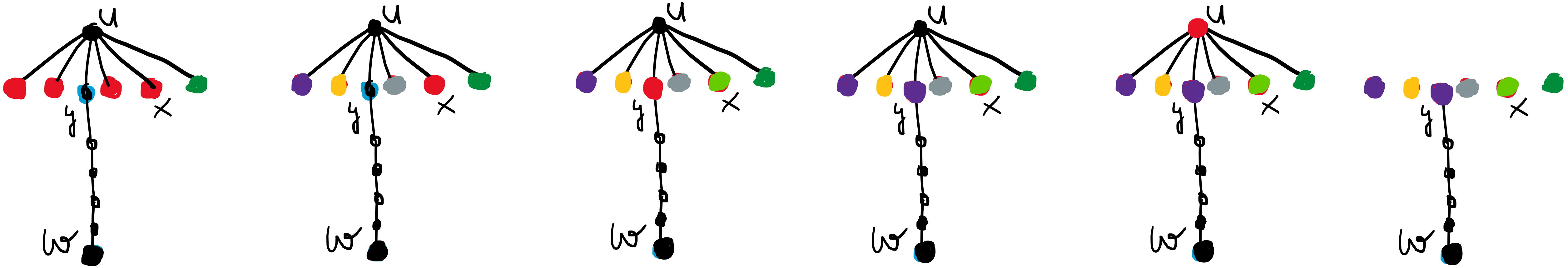
$\lesssim \#P$ steps

1 step

Proof sketch of Key Lemma

Goal recolour u to ●

Can always recolour along shortest w,u path P



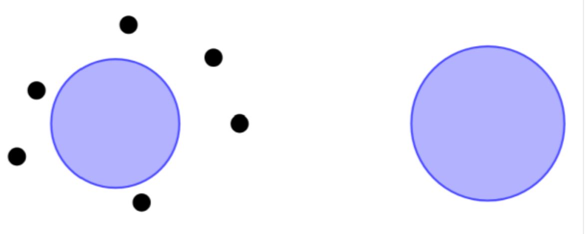
$\Rightarrow$ need $\lesssim 3n$ steps to recolour u

$\Rightarrow$ need $\lesssim \sum_{i=1}^{n} 3i \approx \frac{3}{2}n^2$ steps to recolour all vertices

$\square$

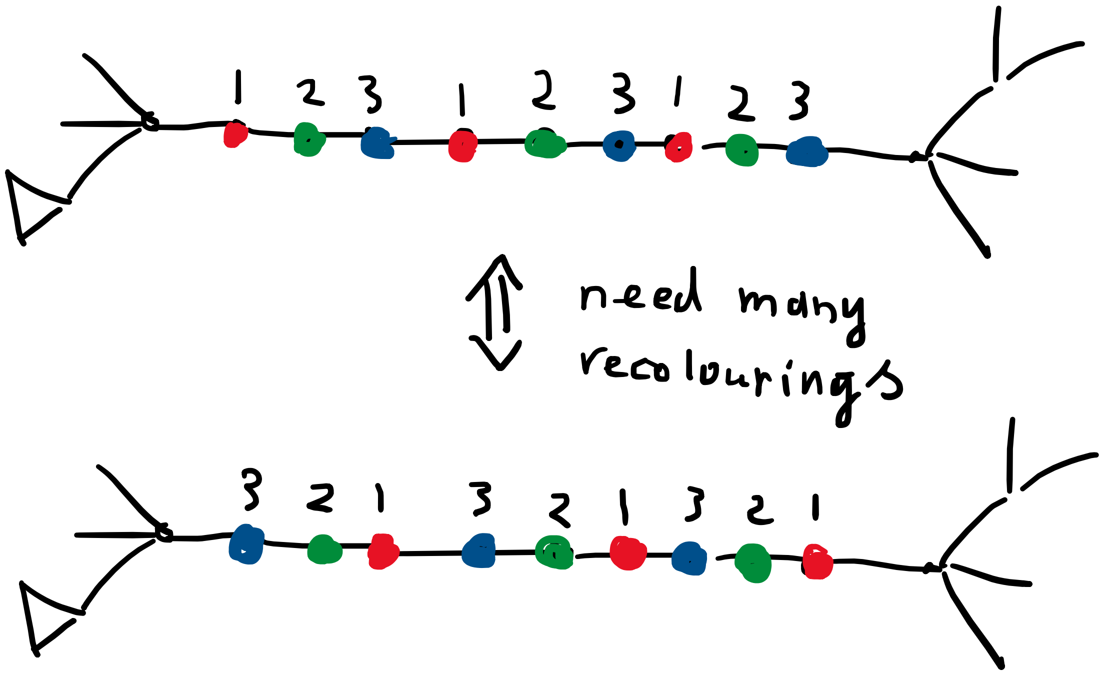
Remark diameter $O(n^2)$ is optimal:

(Both in Key Lemma and Main Theorem)



Remark diameter $O(n^2)$ is optimal:

Proof G could have long induced path P_t of degree 2 vertices with $t = \Omega(n)$, and list $\{1,2,3\}$ on each vertex, so that



$$\mathrm{diam}\,(\hat{R}_L(G)) \geq \mathrm{diam}\,(R_3(P_t)) \geq \frac{1}{4}t^2 = \Omega(n^2)$$

$\square$

Key Lemma

If $|L(w)| \geq \deg(w)+2$ for some w

and $|L(v)| \geq \deg(v)+1$ for all other v,

then $\operatorname{diam}(R_L(G)) = \Theta(n^2)$.

Key Lemma variant

If $|L(w)| \geq \deg(w)+2$ for some w

and $|L(v)| \geq \deg(v)+1$ for all other v,

and minimum degree ≥ 3

then $\operatorname{diam}(R_L(G)) = O(\Delta \cdot n)$.

Key Lemma variant

If $|L(w)| \geq \deg(w)+2$ for some w

and $|L(v)| \geq \deg(v)+1$ for all other v,

and minimum degree ≥ 3

then $\mathrm{diam}\,(R_L(G)) = \Theta(\text{average degree} \cdot n)$.

Key Lemma variant ?

If $|L(w)| \geq \deg(w)+2$ for some w
and $|L(v)| \geq \deg(v)+1$ for all other v,
and minimum degree ≥ 3
then $\operatorname{diam}(R_L(G)) = O(n)$?

Open Problems

„easy"

$$\left.\begin{array}{l} |L(w)| \geq \deg(w)+2 \\ |L(v)| \geq \deg(v)+1 \quad \forall v \neq w \\ \text{\& minimum degree} \geq 3 \end{array}\right\} \implies \operatorname{diam}\left(R_L(G)\right) = O(n)\,?$$

$$\left.\begin{array}{l} |L(v)| \geq \deg(v)+1 \quad \forall v \\ \text{\& minimum degree} \geq 3 \end{array}\right\} \implies \operatorname{diam}\left(\hat{R}_L(G)\right) = O(n)\,?$$

$$\left.\begin{array}{l} |L(v)| \geq \deg(v)+1 \quad \forall v \\ \text{\& no path of } t \text{ consecutive} \\ \text{degree-2 vertices} \end{array}\right\} \implies \operatorname{diam}\left(\hat{R}_L(G)\right) = O((t+1)\cdot n)\,?$$

hard

Summary

If $|L(v)| \geq \deg(v)+1$, then

1. $\exists$ exponentially many L-colourings
2. (G,L) admits a fractional list-packing
3. the reconfiguration graph of L-colourings is essentially connected, with diameter $O(n^2)$.

Bonus

$\boxed{R_L(\mathcal{G}) \text{ connected}}$

$\Downarrow$

"Glauber dynamics" (a random walk on $R_L(\mathcal{G})$ + loops)

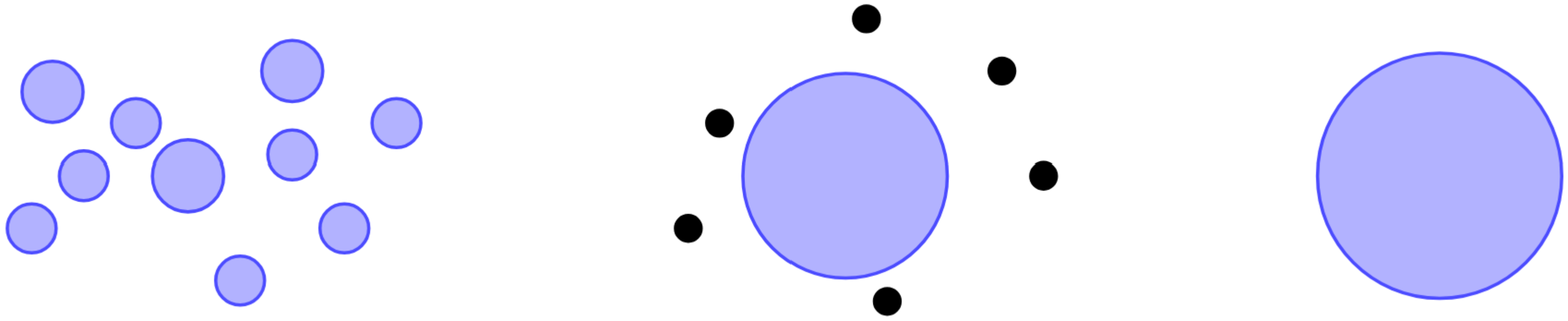
converges to uniform distribution over all L-colourings.

$\Downarrow$

can sample uniformly random L-colouring &

can approximately count # L-colourings.

Thank you for your attention!



Reconfiguration of list colourings, arxiv:2505.08020
Fractional list packing for layered graphs, arxiv:2410.02695
List packing number of bounded degree graphs, arxiv:2303.01246
Optimally reconfiguring list and correspondence colourings, arxiv:2204.07928

Slides available at woutercvb.github.io

Thm (Cambie, CvB, Cranston, 2023)

$$\operatorname{diam}(R_{\Delta+2}(G)) \le 2n$$

Conj. $$\operatorname{diam}(R_{\Delta+2}(G)) = n + \mu$$ where $\mu =$ maximum size of a matching of G.

Thm (De Meyer, 2025+)

Conjecture is True for subcubic graphs & complete multipartite graphs.

Glauber dynamics

Initialize with any L-colouring, then repeat:

(i) Choose uniformly random vertex v.

(ii) Choose " " random colour $x \in L(v)$,

(iii) Recolour v to x if it yields proper L-colouring.
o/w keep current colouring.

When $R_L(G)$ is connected, this (irreducible symmetric) <u>Markov chain</u> converges to the uniform distribution over all L-colourings. Hence this process can be used to sample a $\approx$ uniformly random L-colouring.